Mathematical Reviews

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ANALYSIS

Fourier Series and Integrals, Theory of Approximation

Tschakaloff, Ljubomir. Trigonometrische Polynome mit einer Minimumeigenschaft. Ann. Scuola Norm. Super. Pisa (2) **9**, 13–26 (1940). [MF 1750]

The classical proof of the prime number theorem [see, for example, Landau, Handbuch der Lehre von der Verteilung der Primzahlen, vol. I, pp. 245–258, and vol. II, pp. 891–893] suggests the following problem. Let $g(\varphi)=a_0+a_1\cos\varphi+\cdots+a_n\cos n\varphi$ denote a trigonometrical cosine polynomial of order not greater than n, where $a_0\geqq 0$, $a_1\geqq 0$, $\cdots$, $a_n\geqq 0$; $a_1>a_0$. What is the minimum P_n of the expression $g(0)/(a_1-a_0)$? Landau [Ann. Scuola Norm. Super. Pisa **2**, 209–210 (1933); **5**, 141 (1936)] showed that $P_2=7$, $P_3=P_4=P_5=6$. These results had previously been established by Tschakaloff [Yearbook of the University of Sofia, **19** (1923) in Bulgarian], who had also found the values of P_6, P_7, P_8, P_9. The proof of the latter results is reproduced in the reviewed paper: $P_6=5.9298\cdots$, $P_7=P_8=P_9=5.9052\cdots$. *A. Zygmund* (Cambridge, Mass.).

König, Bedřich. Trigonometrische Entwicklung von $\mathfrak{K}(w,x,s)=\sum_{n=0}^{\infty}e^{2nx\pi i}/(w+n)^s$. Časopis Pěst. Mat. Fys. **69**, 1–7 (1939). (Czech. German summary) [MF 1935]

The author computes the Fourier coefficients of the expression

$$\Phi(w,s,x,z)=\sum_{n=0}^{\infty}\frac{e^{2x[n+(z/\pi)]\pi i}}{\left(w+\dfrac{z}{\pi}+n\right)^s},\qquad -\pi<z<\pi,$$

considered as a function of z, and by specification of the variable z, as well as of the parameters w, s, x, obtains a number of additional formulae. *A. Zygmund.*

Denjoy, Arnaud. Exemples de séries trigonométriques non sommables. C. R. Acad. Sci. Paris **210**, 94–97 (1940). [MF 1241]

The problem of calculating the coefficients of a convergent trigonometrical series with given sum leads to the more general problem of finding a continuous function $F(\theta)$, knowing its generalized second derivative $f(\theta)$, which is connected to F by the relations $F(\theta+u)+F(\theta-u)-2F(u)=u^2R(\theta,u)$, $\lim_{u\to 0}R(\theta,u)=f(\theta)$. The author has defined a method for calculating the functional $V(F,x,y,z)=\sum_{x,y,z}F(x)(z-y)$, when f is given [C. R. Acad. Sci. Paris **172**, 653, 833, 903, 1218 (1921); **173**, 127 (1921)]. This is effected by a well-ordered enumerable sequence of applications of five distinct kinds of operation (a process analogous to that used in the constructive theory of the Denjoy integral). The object of this paper is to show by examples that, if f belongs to the most general class of functions expressible as convergent trigonometrical series, then all five kinds of operation may be required in calculating F. The series is constructed which converges to a function f, whose points of non-summability (non-integrability L) form a perfect set P, while F has infinite derivatives at the ends of every interval contiguous to P. *L. S. Bosanquet* (Bangor).

Izumi, Shin-ichi and Kawata, Tatsuo. Notes on Fourier series, (X). Summability. Tôhoku Math. J. **46**, 154–158 (1939). [MF 1178]

Extending a result of Zalcwasser [Studia Math. **6** (1936)], the authors prove that, if $S_n(x)$ is the nth partial sum of the Fourier series of $f(x)$, then

$$\lim_{n\to\infty}\frac{1}{n}\{S_{1^k}(x)+S_{2^k}(x)+\cdots+S_{n^k}(x)\}=f(x)$$

almost everywhere, for any positive integer k.
E. C. Titchmarsh (Oxford).

Wolf, František. On summable trigonometrical series: an extension of uniqueness theorems. Proc. London Math. Soc. **45**, 328–356 (1939). [MF 248]

Given the series

$$\sum (a_n\cos nx+b_n\sin nx), \tag{1}$$

form

$$S_k(x,\omega)=\sum_{n<\omega}\left(1-\frac{\omega}{n}\right)(a_n\cos nx+b_n\sin nx),$$

$$\bar{S}_k(x,\omega)=\sum_{n<\omega}\left(1-\frac{\omega}{n}\right)(a_n\sin nx-b_n\cos nx).$$

Let $s(x)$, $S(x)$, $\bar{s}(x)$, $\bar{S}(x)$ be the inferior and superior limits of these sums as $\omega\to\infty$. Supposing that $s(x)$ and $S(x)$ are finite in $(a,b)-H$ and P-integrable, where H is an enumerable point set, and that $S_k(x,\omega)=o(\omega)$ in H, the author shows (in Theorem C) the existence of a reducible set E (derived set enumerable) in (a,b) such that (1), in any interval in (a,b) not containing points of E, is a restricted Fourier series, that is, behaves as the Fourier series of $S(x)$ as far as (C,k)-summability is concerned. From Theorem H appears that at an isolated point x_0 of E the series (1) differs from the Fourier series of $S(x)$ by a singular component of the form

$$\sum_{2m<k}A_m\sum_{1}^{\infty}n^{2m}\cos n(x-x_0)+\sum_{2m+1<k}B_m\sum_{1}^{\infty}n^{2m+1}\sin n(x-x_0).$$

These series are summable (C,k) to zero, everywhere in the case of the sine series and for $x\neq x_0$ in the case of the cosine series. Theorem F states that if $k>0$ and $s(x)=S(x)$ in a set of positive measure and are L-integrable and finite in (a,b), then $\bar{S}(x)=\bar{s}(x)$ almost everywhere in (a,b), $S_k(x,\omega)=o(\omega)$ outside of E and in the isolated points of E, $\bar{S}_k(x,\omega)=O(\omega^2)$. Under the assumptions of Theorem C plus $\bar{S}_k(x,\omega)=o(\omega^2)$ for all x in (a,b), the set E is void, and, if

$(a, b)=(0, 2\pi)$, the series is the Fourier series of $S(x)$ and $S(x)=s(x)$ almost everywhere. The first half of the paper is given up to a discussion of properties of a function whose $(2m)$th symmetrical de la Vallée Poussin derivative is given.
E. Hille (New Haven, Conn.).

Beurling, Arne. Ensembles exceptionnels. Acta Math. 72, 1–13 (1939). [MF 1293]

Let $f(\theta)\epsilon L_2(0, 2\pi)$ and periodic of period 2π. Let S be the subspace of L_2 for which

$$S(f)=(2\pi^2)^{-1}\int_0^\infty\int_0^{2\pi} |f(\theta+t)-f(\theta-t)|^2 t^{-2}d\theta dt<\infty.$$

It is easily found that, if

$$f(\theta)\sim\frac{a_0}{2}+\sum_1^\infty(a_n \cos n\theta+b_n \sin n\theta),$$

then $S(f)=\sum n(|a_n|^2+|b_n|^2)$. Let E_f be the set of points where the limit

$$\lim_{h\to 0}(2h)^{-1}\int_{x-h}^{x+h} f(x)dx$$

does not exist or is not finite. The author uses the definition of (logarithmic) capacity $C(O)$ of open sets O given by de la Vallée Poussin [Ann. Inst. H. Poincaré 2, 226 (1932)] and defines the outer capacity $\bar{C}(E)$ of any set E by $\bar{C}(E)=\inf_{O\supset E} C(O)$. With this notation he proves that for a function $f\epsilon S$ the set E_f coincides with the set where the Fourier series of f diverges; the set E_f is of outer capacity zero. The author points out that only the last statement is essentially new. The following lemma, which is of independent interest, plays an important role in the proof. Let

$$F(r, \theta)=\int_0^r\left|\frac{\partial}{\partial r}f(r, \theta)\right|dr,$$

$$f(r, \theta)=\frac{a_0}{2}+\sum_1^\infty r^n(a_n \cos n\theta+b_n \sin n\theta).$$

Then, wherever $f\epsilon S$ so does F, and $S(F)<S(f)$. Among various applications to the theory of functions of a complex variable we cite the following one. Let $f(z)$ be meromorphic in the unit circle and let $f(z)$ map the unit circle into a Riemann surface of finite spherical area $\iint_{|z|<1}|f'|^2(1+|f|^2)^{-2}dxdy$. Then the radial limit $f(\theta)$ of $f(z)$ exists everywhere except on a set E of outer capacity zero. *J. D. Tamarkin.*

Lévy, Paul. Sur quelques problèmes actuellement irrésolus et sans doute insolubles dans les théories des séries et des intégrales de Fourier. J. École Polytech. 145, 179–194 (1939). [MF 1637]

The problems with which the author is chiefly concerned are those of finding necessary and sufficient conditions for a function $\varphi(x)$ to have an absolutely convergent Fourier series, for it to be the Fourier-Stieltjes transform of a distribution function, or for it to be the Fourier transform of a function of $L(-\infty, \infty)$. For the second and third problems, necessary and sufficient conditions (associated with the names of Bochner, Khintchine, Cramér) are known, but the author regards them as unsatisfactory because they are not expressed in terms of the function $\varphi(x)$ alone (as, for example, the Riesz-Fischer theorem is); they either involve transforms of $\varphi(x)$ or demand that $\varphi(x)$ be representable by certain formulas. The author suggests that for all these problems there probably are no "simple" necessary and sufficient conditions, and that it would be interesting to have a precise theorem to this effect (analogous to the theorem that elliptic integrals are not expressible in terms of the elementary functions). The paper concludes with a seven-page "Note sur l'historique du théorème de la Faltung," in which the author discusses the history of the Faltung relations for Fourier integrals, Laplace integrals and Fourier series and the history of their application to the solution of integral equations. Various cases of the theorems are attributed to the author himself, to Doetsch and (with some hesitation) to Horn. The author has failed, however, to notice that the Faltung theorem for Fourier transforms in L^2, which he credits to Doetsch [1936], was explicitly stated and extensively applied at least as early as 1930 by Wiener [Acta Math. 55, 126 (1930); The Fourier Integral and Certain of Its Applications, 1933, pp. 70–71].
R. P. Boas, Jr. (Durham, N. C.).

Tchelidze, W. G. A necessary and sufficient condition that the double Fourier-Lebesgue series of a function belonging to $L^{(2)}$ should be convergent. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 6, 97–124 (1939). (Russian. English summary) [MF 2000]

Lusin proved [C. R. Acad. Sci. Paris 156, 1655–1658 (1913)] that a necessary and sufficient condition that the Fourier series of a function $f(x)\epsilon L^2$ should converge almost everywhere is that

$$(*)\qquad \lim_{n\to\infty}\int_0^\pi \frac{g(x+\alpha)-g(x-\alpha)}{\alpha}\cos n\alpha d\alpha=0$$

for almost every x, where $g(x)$ denotes the conjugate function of $f(x)$ (by the Riesz-Fischer theorem, g is of integrable square) and the integral $\int_0^\pi$ is defined as $\lim_{\epsilon\to+0}\int_\epsilon^\pi$.

In the present paper the Lusin theorem is extended to double Fourier series

$$(**)\qquad \sum_{m,n=0}^\infty (a_{mn} \cos mx \cos ny+b_{mn} \sin mx \cos ny +c_{mn} \cos mx \sin ny+d_{mn} \sin mx \sin ny)$$

of functions $F(x, y)$ of the class L^2. The author defines the function $G(x, y)$ conjugate to $F(x, y)$ as the function whose Fourier series is

$$\sum_{m,n=1}^\infty (d_{mn} \cos mx \cos ny-c_{mn} \sin mx \cos ny -b_{mn} \cos mx \sin ny+a_{mn} \sin mx \sin ny)$$

(here again, by the Riesz-Fischer theorem, $G\epsilon L^2$). Then a necessary and sufficient condition that the series (**) should converge almost everywhere is that a set of conditions analogous to (*) and containing the function $G(x, y)$ should be satisfied almost everywhere. The main condition is rather complicated. *A. Zygmund* (Cambridge, Mass.).

González Domínguez, Alberto. The representation of functions by Fourier integrals. Duke Math. J. 6, 246–255 (1940). [MF 1556]

The author obtains necessary and sufficient conditions for the representation of a complex-valued function $f(t)$, bounded in $(-\infty, \infty)$, in the forms

$$(g)\qquad f(t)=\int_{-\infty}^\infty e^{ixt}g(x)dx,$$

$$(G)\qquad f(t)=\int_{-\infty}^\infty e^{ixt}dG(x),$$

with $g(x)$ and $G(x)$ belonging to various classes of functions. These conditions are of the same kind as those obtained by Cramér [Trans. Amer. Math. Soc. 46, 191–201 (1939); these Rev. 1, 13 (1940)] for $g(x)\epsilon L(-\infty,\infty)$ and for $G(x)$ of bounded variation in $(-\infty,\infty)$. They involve an auxiliary function $s(t)\epsilon L(-\infty,\infty)$ such that $s(t)=\int_{-\infty}^{\infty}e^{ixt}K(x)dx$, where $K(x)$ is real, nonnegative and $O(|x|^{-1-\alpha})$ as $|x|\to\infty$ $(\alpha>0)$, and $s(0)=1$. (This $K(x)$ is somewhat more specialized than the corresponding function used by Cramér, but more convenient for the method used by the author.) The conditions, like Cramér's, are expressed in terms of the function

$$g(n,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}s(t/n)f(t)dt.$$

Necessary and sufficient conditions for $f(t)$ to have the representation (g) with $g(x)\epsilon L(-\infty,\infty)$ are

$$(*)\qquad \int_{-\infty}^{\infty}|g(n,x)|dx<M,\qquad n=1,2,\cdots,$$

together with the condition that

$$\int_S|g(n,x)|dx<\epsilon,\qquad n=1,2,\cdots,$$

if the measure of the set S is less than $\delta(\epsilon)$. Cramér's conditions were (*) together with the convergence in mean of order one over $(-\infty,\infty)$ of the sequence $\{g(n,x)\}$; the author gives a simple proof of the necessity of the latter condition. Necessary and sufficient conditions for the representation (g) with $g(x)\epsilon L^p(-\infty,\infty)$, $1<p\leqq\infty$, are condition (*) together with

$$\left\{\int_{-\infty}^{\infty}|g(n,x)|^p dx\right\}^{1/p}<M,\qquad n=1,2,\cdots.$$

For type (G), the author establishes the interesting inversion formulas

$$\lim_{n\to\infty}\int_0^x g(n,x)dx=\frac{G(x+)+G(x-)}{2}-\frac{G(0+)-G(0-)}{2},$$

$$\lim_{n\to\infty}\frac{g(n,x)}{n}=K(0)[G(x+)-G(x-)],$$

and consequently obtains necessary and sufficient conditions for the representation (G) with $G(x)$ continuous and of bounded variation. The representation (g) with $g(x)\epsilon L$ and of bounded variation is also discussed. The author indicates how his methods will generalize theorems of Offord and of Verblunsky on the representation of functions $f(t)$ in the form (g), the integral being summable $(C,1)$.

R. P. Boas, Jr. (Durham, N. C.).

Mohanty, Ramanath. The summation of the integral conjugate to the Fourier integral of the finite type and of the conjugate series. J. Indian Math. Soc. 3, 319–329 (1939). [MF 1579]

Let $f(x)$ be Denjoy integrable in $(-\pi,\pi)$,

$$a_s=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos st\,dt,\quad b_s=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin st\,dt.$$

The author considers the integral

$$(*)\qquad \int_0^{\infty}(b_s\cos sx-a_s\sin sx)ds$$

and various conditions for its convergence and summability. Thus, he proves that, if $\psi(t)=f(x+t)-f(x-t)$, $\omega_1(t)=(1/t)\int_0^t\psi(t)dt$ and the integral $\int_0^{\infty}(\omega_1(t)/t)dt$ exists, then (*) converges to the value

$$\frac{1}{\pi}\left[\omega_1(X)+\int_0^{\infty}\frac{\omega_1(t)}{t}dt\right],$$

where $X=\max(\pi+x,\pi-x)$, provided

$$\lim_{n\to\infty}\int_p^{\infty}\frac{\psi(t)}{t}\cos nt\,dt=0,\qquad p=\frac{2m+1}{n}\frac{\pi}{2},$$

and $2m+1$ is fixed and less than n. He shows that an analogous result holds for (C,k) summability of (*), k a positive integer. He also derives further sufficient conditions for (C,k) summability of (*) and for (Riesz, k) summability of the conjugate of the Fourier series of $f(t)$ in case $f\epsilon L$.

J. D. Tamarkin (Providence, R. I.).

Boas, R. P., Jr. Remarks on a theorem of B. Lewitan. Rec. Math. N.S. [Mat. Sbornik] 5 (47), 185–188 (1939). (English. Russian summary) [MF 1428]

In order to establish a connection between a recent theorem of B. Lewitan and certain known results the author proves the following theorem: Let $f(x)$ be a bounded measurable function defined on $(-\infty,\infty)$ and let

$$\Phi(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)\frac{e^{-ixt}-L_2(x,t)}{-t^2}dt,$$

where

$$L_2(x,t)=\begin{cases}1-ixt, & |t|\leqq 1,\\ 0, & |t|>1.\end{cases}$$

Then $\Phi(x)$ is linear in $x<-A$ and in $x>A$ if and only if $f(x)$ is an entire function of exponential type A.

N. Levinson (Cambridge, Mass.).

Ruelle, U. L'integrale di Fourier per funzioni di più variabili. Ist. Elettrotec. Accad. Navale Livorno, no. 157, 47 pp. (1940). [MF 1856]

The author investigates the representation of functions of two variables by means of the Fourier integral. Functions of one variable may be represented by the single or double Fourier integral. In the case of two variables we have correspondingly double or quadruple integrals. The tests for the convergence of such integrals are the same as the tests for the convergence of double Fourier series, provided that the function considered satisfies certain conditions at infinity.

A. Zygmund (Cambridge, Mass.).

Kaluza, Theodor, jun. Untersuchung fastperiodischer Funktionen mittels äquidistanter Zahlenmengen. J. Reine Angew. Math. 181, 153–176 (1939). [MF 1418]

The author defines the "Periodizitätsgrenze" of an almost periodic function $f(x)$ as the lower bound of the values of ϵ such that the set of ϵ-translation numbers of $f(x)$ contains a complete arithmetic progression $x_0+n\omega$ $(n=0,\pm1,\pm2,\cdots;$ $\omega\neq0)$. He then shows by elementary methods that a necessary and sufficient condition that $f(x)$ be limit periodic is that its Periodizitätsgrenze be zero. He gives a number of applications of these ideas, and in particular shows that the uniqueness and approximation theorems and the Parseval equation for limit periodic functions can all be proved by elementary considerations.

R. H. Cameron.

Sirvint, Georges. Sur un théorème de Vladimir Bernstein. Rec. Math. N.S. [Mat. Sbornik] 6 (48), 175–184 (1939). (French. Russian summary) [MF 1442]

In his book [Leçons sur les progrès récents de la théorie des séries de Dirichlet, Collection Borel, 1933] V. Bernstein gives a proof of the following theorem: If δ is the index of condensation of the sequence λ_n, there exists coefficients a_n such that for the resulting Dirichlet series $\sum a_n \exp(-\lambda_n s)$ the difference $C-S$ is not less than δ. C is the abscissa of convergence and S the abscissa of "ultraconvergence serrée." The present author calls attention to the fact that Bernstein's proof is valid only in the case $\delta<\infty$. He supplies a proof of the theorem for $\delta=\infty$, and gives an example showing that Bernstein's construction may actually break down for $\delta=\infty$.

F. Bohnenblust (Pasadena, Calif.).

Wintner, Aurel. Über die Spektra der Toeplitzschen D-Formen. Monatsh. Math. Phys. 48, 147–152 (1939). [MF 634]

O. Toeplitz [Amer. J. Math. 60, 880–888 (1938)] associates to each formal ordinary Dirichlet series $\sum c_n n^{-s}$ the triangular matrix $D=\|d_{m,n}\|$, where $d_{m,n}=c_{n/m}$, if n/m is an integer, and zero otherwise. He proves that the matrix is bounded in the sense of Hilbert if, and only if, the Dirichlet series converges for $\sigma>0$ $(s=\sigma+it)$ to a bounded function $f(s)$. The present author pursues further the study of the interdependence between the matrix D and the function $f(s)$. He shows that the spectrum of the bounded matrix D is the closure of the set of all complex numbers assumed by the function $f(s)$ in the half-plane $\sigma>0$. (The spectrum of D is defined as the set of all complex numbers λ for which the matrix $\lambda\cdot I-D$ does not possess a unique inverse.)

F. Bohnenblust (Princeton, N. J.).

Avakumović, Vojislav G. Bemerkungen über Laplacesche Integrale, deren Wachstum von Exponentialcharakter ist. I und II. Math. Z. 46, 62–69 (1940). [MF 1478, 1479]

Let (a) $A(u)$ be non-decreasing and let (b) $J(x)=\int_0^\infty e^{-xu}dA(u)$ be convergent for $x>0$. In 1937 [in an unpublished paper], H. R. Pitt showed that, if

$$(1)\qquad J(x)\sim M\exp[1/x],\qquad \text{as } x\to 0,$$

then

$$(2)\qquad \exp[2u^{\frac{1}{2}}-o(u^{\frac{1}{2}})]<A(u)<o(\exp[2u^{\frac{1}{2}}])$$

[for a proof, cf. W. T. Martin and N. Wiener, Duke Math. J. 4, 384–392 (1938)]. V. Avakumović and J. Karamata [Math. Z., 345–356 (1936)] showed that, under the stronger condition that (a′) $A(u)\exp[-2u^{\frac{1}{2}}]$ is non-decreasing, then (b) and (1) imply the sharper result

$$(3)\qquad A(u)\sim\frac{M}{2\pi^{\frac{1}{2}}}u^{-\frac{1}{4}}\exp[2u^{\frac{1}{2}}],\qquad \text{as } u\to\infty.$$

By varying still further the condition (a), but keeping (b) and (1) fixed, the author proves in I of the present paper a very general and sharp result which includes both these extreme cases. The proof is based upon Wiener's general Tauberian theorem. In II the author shows that one can still conclude (3) under (a) if one passes to the complex-plane; more precisely, let (a) hold and let (b′) $J(s)=\int_0^\infty e^{-su}dA(u)$ be convergent for $\Re(s)>0$, then, if $J(s)-M\exp[1/s]$, $M\neq 0$, is uniformly bounded in a convex region whose boundary has contact of the first order with the imaginary axis at $s=0$, the theorem states that (3) follows.

W. T. Martin (Cambridge, Mass.).

Boas, R. P., Jr. and Widder, D. V. An inversion formula for the Laplace integral. Duke Math. J. 6, 1–26 (1940). [MF 1538]

Led by the observation that the product of two Laplace transformations becomes formally a Stieltjes transformation for which an inversion method was previously given by Widder [Trans. Amer. Math. Soc. 43 (1938)] the authors derive new inversion formulae for the Laplace transformations

$$(1)\quad f(s)=\int_0^\infty e^{-st}d\alpha(t),\qquad (2)\quad f(s)=\int_0^\infty e^{-st}\varphi(t)dt.$$

The inversion of (2) is as follows: If the integral (2) converges for $x>x_0$, then for each $c>x_0$ we have

$$\lim_{k\to\infty}\frac{e^{ct}t^{k-1}}{k!(k-2)!}\int_0^\infty\frac{\partial^k}{\partial u^k}(u^{2k-1}e^{-tu})f(u+c)du=\varphi(t)$$

for almost all t in $(0,\infty)$, in particular at every point of continuity of $\varphi(t)$. The corresponding formula for (1) is slightly more complicated. These inversion formulae are used to derive new sets of conditions on $f(x)$ in order that $f(x)$ admit a representation of the type (1) or (2) with functions $\alpha(t)$ and $\varphi(t)$ of the following types: (i) $\varphi(t)$ bounded; (ii) $\varphi(t)\epsilon L^p$ $(p>1)$; (iii) $\alpha(t)$ of bounded variation; (iv) $\alpha(t)$ increasing. These new conditions are shown to be equivalent to the known conditions for such representations. Thus new proofs of those known conditions are given.

I. J. Schoenberg (Waterville, Me.).

Crum, M. M. Some inversion formulae. Quart. J. Math., Oxford Ser. 11, 49–52 (1940). [MF 1866]

If $K_\nu(z)$ is the known Bessel function of imaginary argument then for $a>0$ the operation

$$F(\xi)=\frac{1}{\pi}\int_{-\infty}^\infty e^{\frac{1}{2}\pi(\xi+\eta)}K_{i(\xi+\eta)}(a)G(\eta)d\eta$$

in $L_2(-\infty,\infty)$ is its own inverse. However, the inversion of

$$F(\xi)=\frac{1}{\pi}\int_{-\infty}^\infty K_{i(\xi+\eta)}(ia)G(\eta)d\eta$$

requires replacing a by $-a$.

S. Bochner.

Doetsch, Gustav. Die Eigenwerte und Eigenfunktionen von Integraltransformationen. Math. Ann. 117, 106–128 (1939). [MF 1387]

This paper is devoted to a discussion of the well-known transformation $\psi=\mathfrak{T}_\chi(\varphi)$ defined by

$$\int_0^x\psi(t)dt=\int_0^\infty\frac{\chi(xt)}{t}\varphi(t)dt,$$

in the case in which the transformation takes every function of $L_2(0,\infty)$ into a function of the same space. The first half of the paper gives a reformulation of known results: the use of the special Mellin transformation

$$f^*(y)=\text{l.i.m.}\int_{1/a}^a t^{-\frac{1}{2}+iy}f(t)dt$$

to reduce the equation to its algebraic equivalent

$$\rho^*(y)\psi^*(y)=\vartheta^*(y)\varphi^*(-y)$$

[$\vartheta(t)=\chi(t)/t$, $\rho^*(y)=(\frac{1}{2}-iy)^{-1}$], determination of the condition that $\mathfrak{T}_\chi(\varphi)$ be on L_2 to L_2. In § 4 the solutions of

$\varphi=\lambda\mathfrak{T}_x(\varphi)$ are studied. From the algebraic equivalent follows that, if λ is a characteristic value, then

$$D_\lambda(y)=\lambda^2\vartheta^*(y)\vartheta^*(-y)-\rho^*(y)\rho^*(-y)$$

must vanish on a set $\mathfrak{M}_\lambda$ of positive measure symmetric with respect to $y=0$. To every such value corresponds infinitely many characteristic functions $\varphi(y)$ and $\varphi^*(y)$ can be defined as an arbitrary function in L_2 on $\mathfrak{M}_\lambda$, $y>0$. The characteristic values form a countable set, symmetric with respect to $\lambda=0$, $|\lambda|\geqq 1/M$, and are real if the kernel is real. § 5 treats the unitary and involutory cases, the latter already discussed in detail by H. Kober [Proc. London Math. Soc. **44** (1938)]. In the involutory case $\lambda=\pm1$ and $\mathfrak{M}_\lambda$ is the whole y-axis. The author finally discusses solutions of $\bar{\varphi}=\mu\mathfrak{T}_x(\varphi)$. These conjugate values (Spiegelwerte) fill out the circumferences of a countable set of concentric circles $|\lambda|=r_n$ in the λ-plane. Here r_n defines such a circle if $|\vartheta^*(-y)/\rho^*(-y)|=1/r_n$ in a set $\mathfrak{M}$ of positive measure on the y-axis. *E. Hille* (New Haven, Conn.).

Mohan, Brij. A self-reciprocal function. Proc. Edinburgh Math. Soc. (2) **6**, 92–93 (1939). [MF 1521]

The function $x^{\frac{1}{2}}I_0(\frac{1}{4}x^2)K_0(\frac{1}{4}x^2)$ is its own J_0 transform (the notation is that usual in the theory of Bessel functions). *R. P. Boas, Jr.* (Durham, N. C.).

Mohan, Brij. Theorems connecting self-reciprocal functions. Proc. Benares Math. Soc. **1**, 93–96 (1939). [MF 1535]

If $f(x)$ is R_μ (self-reciprocal in the J_μ transform),

$$Q(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)l(s)x^{-s}ds,$$

and $l(s)=l(1-s)$, then

$$(*)\qquad g(x)=i\int_{-\infty}^{\infty}e^{it}Q(e^{it})f(xe^{it})dt$$

is (formally) R_ν. There is a similar result when the arguments of f and Q are interchanged in (*). A formula of Ramanujan leads to the particular case that if $f(x)$ is $R_{3/2}$, so is $g(x)$, if $g(x)$ is defined by (*) with $Q(u)=u^{\frac{1}{2}}$; there is a similar (more complicated) result for R_ν, $\nu>1$. The reasoning is entirely formal; restrictions on $l(s)$ and $f(x)$ sufficient for the validity of the formulas are left for the reader to supply. *R. P. Boas, Jr.* (Durham, N. C.).

Lifschetz, M. On some questions concerning the determinate case of Hamburger's moment problem. Rec. Math. N.S. [Mat. Sbornik] **6** (48), 293–306 (1939). (Russian. English summary) [MF 1359]

Let (*) $\int_{-\infty}^{\infty}t^k d\sigma(t)=s_k$, $k=0, 1, 2, \cdots$, be an indeterminate Hamburger problem of moments, $D_k(x)$ the corresponding set of orthonormal polynomials,

$$E_k(x)=\int_{-\infty}^{\infty}[D_k(x)-D_k(t)](x-t)^{-1}d\sigma(t),$$

$$Q(x)=-1+x\sum_0^\infty E_k(0)D_k(x),\quad V(x)=x\sum_0^\infty D_k(0)D_k(x),$$

$$K(x,t)=\sum_0^\infty D_k(x)D_k(t),\qquad \rho(x)^{-1}=\sum_{k=0}^\infty|D_k(x)|^2.$$

It is well known [M. Riesz, Ark. Mat. Astr. Fys. **17** (1923)] that $\rho(x)^{-1}<e^{\epsilon r}$, $\epsilon=o(1)$, $|x|=r$. It is also known that for any λ in $(-\infty, \infty)$ there exists a solution $\sigma_\lambda(t)$ of the problem of moments, which is a step function with jumps at the roots $\{\xi_k\}$ of the function $Q_\lambda(t)=Q(t)+\lambda V(t)$. The author proves that a necessary and sufficient condition that a function $\phi(t)$ whose square is integrable over $(-\infty, \infty)$ with respect to an arbitrary solution $\sigma(t)$ of (*) could be represented in the form

$$\phi(t)=\sum_0^\infty c_kD_k(t),\qquad \sum c_k^2<\infty,$$

is that

$$\frac{\phi(t)}{Q_\lambda(t)}=\sum_0^\infty\frac{\alpha_k}{t-\xi_k}$$

for at least one value of λ. It is further proved that if $F(x)=\sum x^{2k}/s_{2k}$ then the order of the entire function $F(x)$ does not exceed that of $Q(x)$, and in case they are equal, then the type of $F(x)$ does not exceed that of $Q(x)$. Finally it is shown that, if $\{\eta_k\}$ is a sequence of real numbers such that $\eta_k\to\infty$ and such that the system of equations (**) $\sum\mu_k\eta_k^p=s_p$, $p=0, 1, 2, \cdots$, admits of a positive solution $\mu_0, \mu_1, \cdots$, then the problem of moments (**) is determined provided the exponent of convergence of the sequence $\{\eta_k\}$ is less than $\limsup_{k\to\infty} 2k\log 2k/(\log s_{2k})$.

J. D. Tamarkin (Providence, R. I.).

Boas, R. P., Jr. A completeness theorem. Amer. J. Math. **62**, 312–318 (1940). [MF 1766]

The problem of the completeness, over the interval $-\pi\leqq x\leqq\pi$, of the system of functions e^{2inx}, xe^{2inx} $(n=0, \pm1, \pm2, \cdots)$ may be considered as a special case of the problem of the completeness of the system $e^{i\lambda_n x}$ $(n=0, \pm1, \pm2, \cdots)$ when some of the numbers λ_n coincide. Generalizing the former question, the author shows that, if $G(x)\epsilon L^2(-\pi, \pi)$, then the necessary and sufficient condition that the set of functions e^{2inx}, $G(x)e^{2inx}$ $(n=0, \pm1, \pm2, \cdots)$ should be complete in $L^2(-\pi, \pi)$ is that $G(x+\pi)-G(x)\neq0$ at almost every point of the interval $-\pi<x<0$. Generalizations of this theorem are given and its connections with uniqueness theorems for entire functions of exponential type are stressed [cf. also R. P. Boas, Amer. J. Math. **62**, 319–324 (1940); cf. these Rev. **1**, 212]. *A. Zygmund.*

Boas, R. P., Jr. General expansion theorems. Proc. Nat. Acad. Sci. U. S. A. **26**, 139–143 (1940). [MF 1284]

The author states the theorem: let $\{x_n\}$ and $\{y_n\}$ be two sequences of the normed linear space E. If there is a number λ, $0<\lambda<1$, such that for every sequence $\{a_n\}$ of complex numbers

$$\|\sum_{n=1}^N a(x_n-y_n)\|\leqq\lambda\|\sum_{n=1}^N a_nx_n\|,\quad N=1, 2, \cdots,$$

then, if $\{x_n\}$ is a fundamental set or a base, $\{y_n\}$ has the same property. As a consequence of this theorem it appears that, if, for every sequence $\{a_n\}$, some $p\geqq1$ and all N,

$$\int_0^{2\pi}|\sum_{n=0}^N a_n[z^n-g_n(z)]|^pd\theta\leqq\lambda^p\int_0^{2\pi}|\sum_{n=0}^N a_nz^n|^pd\theta,$$

$z=re^{i\theta}$, $\lambda<1$, then any function $f(z)$ analytic in $|z|<r$ and continuous in $|z|\leqq r$ can be expanded in a series of the functions $g_n(z)$, the series converging uniformly in every circle $|z|=r'<r$. The author states that a considerable number of expansion theorems can be deduced from this result. Detailed proofs of the above results, and of similar other results, are postponed for a subsequent paper.

J. D. Tamarkin (Providence, R. I.).

Obrechkoff, Nikola. Sommation par la transformation d'Euler les séries de Dirichlet, les séries de facultés et la série de Newton. Ann. Univ. Sofia. II. Fac. Phys. Math. Livre 1. 35, 1–156 (1939). (Bulgarian. French summary) [MF 1374]

The author discusses the Euler-Knopp definition of summability E_k, which transforms a given series $\sum_0^\infty a_n$ into $\sum_0^\infty a_n^{(k)}$, where

$$a_n^{(k)}=(q+1)^{-n-1}\sum_{\nu=0}^{\infty}\binom{n}{\nu}q^{n-\nu}a_\nu,\quad q=2^k-1;\, k>0.$$

He applies this method to the Dirichlet series $\sum_0^\infty a_n(n+1)^{-s}$, the factorial series $a_0+\sum_{n=1}^\infty a_n/[(s+1)\cdots(s+n)]$ and the Newton series $a_0+\sum_{n=1}^\infty a_n(s-1)\cdots(s-n)$. In each case it is established that if the series is E_k-summable for some $s=s_0$ then it is E_k-summable for each s with $\Re s>\Re s_0$, the same being true when E_k-summability is replaced by $\bar{E}_k$, the absolute E_k-summability. This shows the existence of abscissas of E_k-summability σ_k, and of absolute E_k-summability $\bar{\sigma}_k$, which are given by the formulas

$$\sigma_k=\limsup_{n\to\infty}(\log n)^{-1}\log|A_0+\cdots+A_n|,\qquad \text{if}\quad \sigma_k\geqq 0,$$

$$\sigma_k=\limsup_{n\to\infty}(\log n)^{-1}\log|A_n+A_{n+1}+\cdots|,\qquad \text{if}\quad \sigma_k<0,$$

and analogous formulas with A_j replaced by $|A_j|$ for $\bar{\sigma}_k$. Here

$$A_n=(q+1)^{-n-1}\sum_{\nu=0}^{\infty}\binom{n}{\nu}q^{n-\nu}a_\nu$$

in the case of the Dirichlet series, and is given by analogous expressions with a_ν replaced by $a_\nu/(\nu!)$ and by $a_\nu(\nu!)$ in the cases of factorial and Newton series, respectively. It is shown, in the cases of Dirichlet and factorial series, that $0\leqq\bar{\sigma}_k-\sigma_k\leqq 1$. It is finally shown that, if a factorial series is Borel summable for $s=s_0$, it will be also Borel summable for each s with $\Re s>\Re s_0$. *J. D. Tamarkin.*

Menchoff, D. Sur les multiplicateurs de convergence pour les séries de polynômes orthogonaux. Rec. Math. N.S. [Mat. Sbornik] 6 (48), 27–52 (1939). (French. Russian summary) [MF 1433]

The author proves the following theorem. There exists a set of polynomials $P_n(x)$ $(n=1, 2, \cdots)$, orthonormal and uniformly bounded on an interval $[a, b]$, which possess the following property: for each positive function $W(n)$ satisfying the condition $W(n)(\log n)^{-2}\to 0$ as $n\to\infty$, it is possible to determine a sequence of real constants c_n such that $\sum c_nP_n(x)$ diverges everywhere on $[a, b]$, while the series $\sum W_nc_n^2$ converges. It follows in particular that there exist functions of class $L_2[a, b]$ for which the Fourier series $\sum c_nP_n(x)$ diverges on $[a, b]$. The set of polynomials can be replaced by a set of functions of a linear family F possessing the following property: for each function $f(x)$ continuous on $[a, b]$ and satisfying $f(a)=f(b)$ and each $\epsilon>0$, there exists a function $\psi(x)\epsilon F$ such that $|f(x)-\psi(x)|<\epsilon$ on $[a, b]$. The proof is based on a body of four lemmas, the proof of which is rather long and can not be reviewed here.

J. D. Tamarkin (Providence, R. I.).

Peebles, Glenn. Some generalizations of the theory of orthogonal polynomials. Duke Math. J. 6, 89–100 (1940). [MF 1545]

Let the "weight function" $\rho(x)$ be such that $M\rho'=L\rho$, where M is a quadratic and L a linear function. The polynomials $\rho^{-1}(d/dx)^n(M^n\rho)$ are generalizations of the classical polynomials; they do not satisfy, however, the orthogonality relations when $\rho(x)$ is not integrable. Nevertheless, certain properties of the classical polynomials hold true, for instance, the recursion formula and the Christoffel-Darboux identity. The general hypergeometric (Jacobi) polynomials are a special case; for this case an equiconvergence theorem is proved. Finally, the construction of a system of polynomials orthogonal with respect to a "weight function" which oscillates in the given interval of orthogonality is described. *G. Szegö* (Stanford University, Calif.).

Banerjee, D. P. On the expansion of a function in a generalized Neumann series. Quart. J. Math., Oxford Ser. 10, 261–265 (1939). [MF 1036]

The author gives a method of expanding certain classes of integral functions $f(z)$ in series of the form

$$(1)\qquad f(x)=\sum_{n=0}^{\infty}a_nz^nJ_n(z),\quad f(z)=\sum_{n=0}^{\infty}a_nJ_n(\lambda z)I_n(\mu z).$$

He first computes the Laplace transform

$$\phi(w)=\int_0^\infty f(z)e^{-wz}dz.$$

The function $\phi(w)$ is expanded in a special way into an infinite series and $f(z)$ is found by means of the inversion formula for the Laplace integrals and term by term integration. For suitable expansions of $\phi(w)$ the resulting series for $f(z)$ is of the type (1). Among other formulas the author obtains

$$\cos z=1-\sum_{n=1}^{\infty}\frac{(\frac{1}{2}z)^nJ_n(z)}{n!(2n-1)}.$$

C. S. Meijer (Groningen).

Favard, J. Sur les meilleurs procédés d'approximation. Ann. Chaire Phys. Math. Kieff 4, 159–168 (1939). (Ukrainian and French) [MF 1561]

The author has shown [Bull. Sci. Math. (2) 61, 209–224, 243–256 (1937)] that for $n=1, 2, \cdots$ there exist constants $\delta_k^{(m)}(n)$ such that the sums

$$S_m(x)=\frac{a_0}{2}+\sum_{k=1}^{m-1}\delta_k^{(m)}(n)(a_k\cos kx+b_k\sin kx)$$

furnish the best approximation to functions $f(x)$, of period 2π, having bounded nth derivatives; in the definition of $S_m(x)$, a_k and b_k are the Fourier coefficients of $f(x)$. The approximation is "best" in the sense that for a certain $K(n)$, and all $f(x)$ with $|f^{(n)}(x)|\leqq M_n$, $|f(x)-S_m(x)|\leqq K(n)m^{-n}M_n$ for all m and x, while, for each m, equality occurs for some $f(x)$ and some value of x. In this paper the author determines explicitly a set of constants $\delta_k^{(m)}(n)$ having the property described, namely

$$\delta_k^{(m)}(n)=\frac{1}{(n-1)!}\left(\frac{k\pi}{2m}\right)^n\psi^{(n-1)}\left(\frac{k\pi}{2m}\right),$$

where $\psi(x)=\cot x$ if n is even, $\psi(x)=-\csc x$ if n is odd.

R. P. Boas, Jr. (Durham, N. C.).

Inoue, Masao. Sur l'approximation des fonctions continues par des fonctions harmoniques (II). Proc. Imp. Acad., Tokyo 15, 211–213 (1939). [MF 1140]

L'auteur revient sur une note antérieure de même titre [Proc. Imp. Acad., Tokyo 15, 177–181 (1939)] pour carac-

tériser autrement sa classe P d'ensembles plans bornés fermés F sur lesquels toute fonction finie continue peut être approchée uniformément par des fonctions harmoniques (dans des domaines contenant F). *M. Brelot.*

Special Functions

Erdélyi, A. Two infinite integrals. Proc. Edinburgh Math. Soc. (2) 6, 94–104 (1939). [MF 1522]

The integrals

$$F(\lambda,\theta)=\frac{1}{2\pi}\int_0^\infty e^{i\lambda\cosh t}\frac{\sin\theta}{\cosh t+\cos\theta}dt,$$

$$G(\lambda,\theta)=\frac{1}{2\pi}\int_0^\infty e^{i\lambda\cosh t}\frac{\cos\frac{1}{2}\theta\cosh\frac{1}{2}t}{\cosh t+\cos\theta}dt,$$

which occur in the theory of diffraction, have been previously considered by F. Kottler [Ann. Physik **71**, 457–508 (1923)]; E. T. Copson and W. L. Ferrar [Proc. Edinburgh Math. Soc. (2) **5**, 159–168 (1938)]; G. N. Watson [ibid. **5**, 173–181 (1938)]; A. Erdélyi [ibid. **6**, 11 (1939)]. Erdélyi now obtains for $F(\lambda,\theta)$ a Fourier series expansion different from Copson and Ferrar's series involving cut Bessel functions and an expansion in ascending powers of $\cos\theta$ whose coefficients involve Lommel's function $s_{\mu,\nu}(\lambda)$. He also shows that the coefficients in Watson's expansions of $F(\lambda,\theta)$ and $G(\lambda,\theta)$ in ascending powers of $\sin\frac{1}{2}\theta$ can be expressed in terms of the confluent hypergeometric function $W_{k,m}(-2i\lambda)$. Lastly he proves that

$$G(\lambda,\theta)=\frac{1}{\pi^{\frac{1}{2}}}e^{-i\lambda\cos\theta}\operatorname{Erfc}\{(-2i\lambda\cos^2\tfrac{1}{2}\theta)^{\frac{1}{2}}\}.$$

E. T. Copson (Dundee).

Varma, R. S. On the polynomial $\pi_n(x)$. Proc. Indian Acad. Sci., Sect. A. 11, 21–22 (1940). [MF 1500]

Angelescu [C. R. Acad. Sci. Roum. **2**, 199–201 (1938)] introduced polynomials $\pi_n(x)$, which, according to B. S. Sastry [Proc. Indian Acad. Sci., Sect. A. **10**, 176–180 (1939)], may be defined as follows: Given a power series

$$\varphi(z)=\sum_0^\infty\frac{a_n}{n!}z^n,$$

let

$$(1-t)^{-1}\exp\left(-\frac{xt}{1-t}\right)\varphi\left(\frac{-t}{1-t}\right)\equiv\sum_0^\infty\frac{\pi_n(x)}{n!}t^n.$$

The Laguerre polynomial $L_n(x)$ is a special case. From the above identity the author deduces two formulae connecting the $L_n(x)$ with the $\pi_n(x)$. *O. Szász* (Cincinnati, Ohio).

Varma, R. S. Some infinite series involving Sonine's polynomial. J. Indian Math. Soc. 3, 330–333 (1939). [MF 1580]

A few particular series of types $\sum b_rT_r^n(x)$ and $\sum b_rx^rT_r^n(x)$ containing Sonine's polynomials

$$T_r^n(x)=\frac{(-)^n}{\Gamma(r+1)\Gamma(n+1)}\,{}_1F_1(-n;r+1;x).$$

Example:

$$\sum_{r=0}^\infty\frac{(-)^r}{(2r)!}x^{2r}T_{2r}^n(x)=\int_0^x t^{-1}T_{-1}^n(t)\operatorname{ber}(2(x-t)^{\frac{1}{2}})dt,\quad n>0.$$

The series are obtained either by taking special values of the parameters in known expansions or by operational calculus. *A. Erdélyi* (Edinburgh).

Palamà, G. Su delle relazioni integrali relative ai polinomi di Laguerre e d'Hermite. Rend. Sem. Mat. Univ. Padova 10, 46–54 (1939). [MF 1747]

Computation of various integrals of the form

$$\int_{-\infty}^\infty e^{-a^2x^2}H_n{}^m(x)H_k(ax)dx$$

involving Hermite polynomials and of related integrals in terms of Laguerre polynomials as, for instance,

$$\int_0^\infty e^{-x}x^\beta L_n{}^{(\alpha)}(x)L_k{}^{(\beta)}(x)dx,$$

as well as mixed integrals involving both types of polynomials. *E. Hille* (New Haven, Conn.).

Feldheim, Ervin. Sur les fonctions génératrices des polynomes de Laguerre et d'Hermite. Bull. Sci. Math. 63, 307–329 (1939). [MF 1626]

The author discusses the relations between the generating function of a set of orthogonal polynomials $\{\omega_n(x)\}$ and the generating function of $\{\omega_n(x)\omega_n(y)\}$ with particular reference to Hermite and Laguerre polynomials. The results so obtained are not new nor are the proofs essentially new, but the underlying general principle is worth a review. Let $\omega_n(x)=a_nx^n+\cdots$ and

$$\int_a^b p(x)\omega_m(x)\omega_n(x)dx=\delta_{mn}\lambda_n.$$

Let

$$K(x,y\ t)=[p(x)p(y)]^{\frac{1}{2}}\sum_{n=0}^\infty\omega_n(x)\omega_n(y)\frac{t^n}{\lambda_n}.$$

Then, formally

$$\sum_0^\infty\frac{a_n}{\lambda_n}\omega_n(x)t^n=[p(x)]^{\frac{1}{2}}\lim_{\epsilon\to0}[p(1/\epsilon)]^{-\frac{1}{2}}K(x,1/\epsilon;\epsilon t).$$

Conversely, if

$$F(x,t)=\sum_0^\infty A_n\omega_n(x)t^n$$

is a given generating function, then

$$A_n\lambda_nt^n=\int_a^b p(x)F(x,t)\omega_n(x)dx,$$

and, if there exists an inverse transformation such that

$$c_n\omega_n(x)=\int_a^b p(t)F(x,t)f(t)t^ndt$$

for some choice of $f(t)$, then

$$\sum_0^\infty c_nA_n\omega_n(x)\omega_n(y)t^n=\int_a^b p(v)F(x,tv)F(y,v)f(v)dv.$$

Thus the problem requires finding the inverse of the transformation T which takes $\omega_n(x)$ into some multiple of t^n. For the case of Hermite and Laguerre polynomials, T is known to be the Gauss and the Hankel transformations, respectively, with known inverses. *E. Hille.*

Feldheim, Ervin. Développements en série de polynomes d'Hermite et de Laguerre à l'aide des transformations de Gauss et de Hankel. I et II. Nederl. Akad. Wetensch., Proc. **43**, 224–248 (1940). [MF 1595]

The series expansions referred to in the title are mainly expansions of different types of Hermite and Laguerre polynomials in terms of each other derived with the aid of the transformations of Gauss and of Hankel. The only infinite series occurring are various types of generating functions including several for products of Hermite polynomials. The paper contains 100 numbered relations between Hermite and Laguerre polynomials of which the following random samples can be quoted as typical of the character of the results:

$$(24'')\qquad H_{2n}(y)=\sum_0^n(-2)^{n-\nu}(n-\nu)!\binom{n}{\nu}^2H_\nu^2(y),$$

$$(30)\qquad \begin{aligned}&H_n^2(x)H_n^2(y)\\&=(-1)^n(n!)^2\sum_0^n\frac{(-1)^k}{(k!)^2}H_{2k}(x)H_{2k}(y)L_{n-k}^{(2k)}(2x^2+2y^2),\end{aligned}$$

$$(44)\qquad H_{2n+1}(y)=2y\sum_0^n(-4)^{n-k}(n-k)!\binom{n}{k}H_{2k}(y).$$

Some of the formulas of the second part will perhaps find a wider range of usefulness as for instance

$$(52\text{a})\qquad \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-t^2}H_m\left(\frac{\nu+t}{\sqrt{2}}\right)H_n\left(\frac{\nu-t}{\sqrt{2}}\right)dt =2^{-\frac{1}{2}(m+n)}H_{m+n}(\nu).$$

E. Hille (New Haven, Conn.).

Feldheim, Ervin. Développements en série de polynomes d'Hermite et de Laguerre à l'aide des transformations de Gauss et de Hankel. III. Nederl. Akad. Wetensch., Proc. **43**, 379–386 (1940). [MF 1914]

Continuation of parts I and II [see the above review]. Of the about 25 additional relations between Hermite and Laguerre polynomials, listed by the author, formula (63), which follows, can serve as a typical sample:

$$2^m n!\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(u+ix)^2-(v+ix)^2}(iu)^{m-n}L_n^{(m-n)}(2uv)\,du\,dv =(-1)^nH_{m+n}(x).$$

A couple of relations between general Whittaker functions stated without proof on p. 381 would seem to be of some interest. They imply that the Hankel transform of order $-\frac{1}{2}$ of the product $u^{-3/4}W_{\lambda,\nu}(u)W_{\mu,\nu}(u)$ is a constant multiple of $u^{-3/4}W_{\lambda+\mu,\lambda-\mu}(2u)$ if $\lambda=(m/2)+\frac{1}{4}$, $\mu=(n/2)+\frac{1}{4}$, $\nu=\pm\frac{1}{4}$, and $m+n$ is an even integer. The inverse is also true.

E. Hille (New Haven, Conn.).

Toscano, Letterio. La trasformazione di Gauss e i polinomi di Hermite. Atti Accad. Sci. Torino **75**, 39–46 (1939). [MF 1883]

A relation between powers of variables gives rise to a corresponding relation between Hermite polynomials in these variables through the "Anti-Gauss" transformation which carries x^n into $H_n(x)$. The following formulas are typical. The identity

$$(a_1s_1+a_2s_2+\cdots+a_ns_n)^m =\sum C^m_{i_1i_2\cdots i_n}a_1^{i_1}a_2^{i_2}\cdots a_n^{i_n}s_1^{i_1}s_2^{i_2}\cdots s_n^{i_n},$$

after ν applications of the inverse of the Gauss transformation, becomes

$$\begin{aligned}&(a_1^2+a_2^2+\cdots+a_\nu^2)^{m/2}H_m\\&\quad\times\left[\frac{a_1x_1+\cdots+a_\nu x_\nu+a_{\nu+1}s_{\nu+1}+\cdots+a_ns_n}{(a_1^2+a_2^2+\cdots+a_\nu^2)^{\frac{1}{2}}}\right]\\&\quad=\sum C^m_{i_1i_2\cdots i_n}a_1^{i_1}a_2^{i_2}\cdots a_n^{i_n}\\&\qquad\qquad\times H_{i_1}(x_1)H_{i_2}(x_2)\cdots H_{i_\nu}(x_\nu)s_{\nu+1}^{i_{\nu+1}}\cdots s_n^{i_n}.\end{aligned}$$

E. Hille (New Haven, Conn.).

Bailey, W. N. On the product of two associated Legendre functions. Quart. J. Math., Oxford Ser. **11**, 30–35 (1940). [MF 1864]

The author's object is to give a direct proof of a formula of the type

$$(\mu^2-1)^{-\frac{1}{2}m}P_p^m(\mu)P_q^m(\mu)=\sum A^m_{r,p,q}P^m_{p+q-m-2r}(\mu),$$

where the coefficients are certain specified quotients of factorials. Assuming $\gamma+\gamma'=\alpha+\beta+\alpha'+\beta'$, the author derives the expansion

$$\begin{aligned}&F(\alpha,\beta;\gamma;x)F(\alpha',\beta';\gamma';x)\\&\quad=\sum_{r=0}^{\infty}\frac{(\gamma'-\beta')_r(\alpha)_r(\beta)_r(\gamma+\gamma'-1)_{2r}}{r!(\gamma)_r(\gamma')_r(\alpha+\beta+\alpha'+r-1)_r(\gamma+\gamma'-1)_r}\\&\qquad\times x^r{}_4F_3\left[\begin{matrix}\alpha+\alpha'+r,\ \alpha'+\beta+r,\ \beta',\ \gamma+\gamma'-1+2r;\\ \gamma'+r,\ \alpha+\beta+\alpha'+2r,\ \gamma+\gamma'-1+r\end{matrix}\ x\right].\end{aligned}$$

If $\gamma=1+\alpha-\beta$, $\gamma'=\alpha'+\beta$, the ${}_4F_3$ reduces to an ordinary hypergeometric function $F(\alpha+\alpha'+2r,1-\beta;\alpha+\beta+\alpha'+2r;x)$ which is expressible in terms of associated Legendre functions. For a suitable choice of the parameters this leads to the desired relation. Other specializations give related formulas in which the exponent $-\frac{1}{2}m$ is replaced by $\frac{1}{2}m$ or the functions of the first kind are replaced by those of the second.

E. Hille (New Haven, Conn.).

Shabde, N. G. On some results involving Legendre and Bessel functions. Proc. Benares Math. Soc. **1**, 55–59 (1939). [MF 1531]

Expressions are given for $\int_{-1}^{1}(1-\mu^2)^{m/2}P_p^m(\mu)P_q^m(\mu)d\mu f(\mu)$, when $f(\mu)=\mu P_t^m(\mu)$ and when $f(\mu)(z-\mu)=1$. In the former case the integral is zero when $p+q+m-t\neq\pm1$ or when $p-q-m-t\neq\pm1$, p, q, m and t being positive integers. The integral $\int_0^1P_p(1-2y^2)P_q(1-2y^2)J_n(2yz)y^{n+1}$ is expressed as a series involving a finite number of Bessel functions in the two cases $n=0,1$. Two other integrals are also expressed in such a form. The paper ends with the values of some simple definite integrals involving the functions $P_{m,n}(x)$, $Q_{m,n}(x)$.

H. Bateman (Pasadena, Calif.).

Shabde, N. G. On some series and integrals involving k_n-functions. J. Indian Math. Soc. **3**, 307–311 (1939). [MF 1577]

Some series are first obtained by replacing t by it in the equations

$$ze^{t-z}=\sum_{n=1}^{\infty}n(t^n+t^{n+1})k_{2n}(x),$$

$$ye^{-(z+t)}J_1(2y)=-\sum_{n=1}^{\infty}(-t)^nk_{2n}(x)/\Gamma(n),$$

wherein $z+zt=2xt=y^2$. Two integrals involving Jacobi polynomials and the Bessel function $J_1(2(xy)^{\frac{1}{2}})$ are then

expressed by series of k-functions and an integral involving products of k-functions and Bessel functions is expressed by means of Lauricella's hypergeometric function of four variables. The product of two k-functions is expressed in two ways by an integral over a finite range and an integral for the k-function is derived from a known integral involving a Bessel function. *H. Bateman* (Pasadena, Calif.).

Kibble, W. F. A Bessel function in terms of incomplete Gamma functions. J. Indian Math. Soc. **3**, 271–294 (1939). [MF 1025]

Let

$$\gamma(s, z)=\int_0^z t^{s-1}e^{-t}dt=z^s e^{-z}\sum_{h=0}^{\infty}\frac{z^h}{s(s+1)\cdots(s+h)}$$

be the incomplete Gamma function. The author investigates which series of the kind

$$z^a e^{pz}\sum_{m=-\infty}^{+\infty} c_m(kz)^{-m}\gamma(\mu+m, kz)$$

satisfy a Bessel differential equation

$$F''+\tfrac{1}{z}F'-\left(K^2+\frac{N^2}{z^2}\right)F=0,$$

and shows for $R(\nu)>-\frac{1}{2}$ the Bessel function of imaginary argument

$$I_\nu(z)=\frac{e^z(2z)^a}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\sum_{-\infty}^{+\infty}\binom{\nu-\frac{1}{2}}{m-a-\frac{1}{2}}(-2z)^{-m}\gamma(\nu+m-a, 2z),$$

where a is arbitrary and

$$\binom{\nu-\frac{1}{2}}{m-a-\frac{1}{2}}=\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu-m+a+1)\Gamma(m-a+\frac{1}{2})}.$$

If $\nu+a$ is an integer, then the simpler formula

$$I_\nu(z)=\frac{e^z(2z)^{-\nu}}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\sum_{m=0}^{\infty}\binom{\nu-\frac{1}{2}}{m}(-2z)^m\gamma(2\nu-m, 2z)$$

is obtained, and this result can be further simplified if 2ν is an odd integer. For earlier results, see G. N. Watson, Theory of Bessel Functions, 1922, chap. 7.

K. Mahler (Manchester).

Bock, Philipp. Über einige Integrale aus der Theorie der Besselschen, Whittakerschen und verwandter Funktionen. Nieuw Arch. Wiskde **20**, 163–170 (1940). [MF 1093]

There is a characteristic relation between two functions which are Hankel transforms of each other, dealt with, for example, by F. Tricomi [Atti Accad. Naz. Lincei (6) **22**, 564–571 (1935)]. In the present paper Tricomi's formula is applied to some pairs of functions, Hankel transforms of each other, which were given by C. S. Meijer [in several papers in the Nederl. Akad. Wetensch., Proc., from 1934 onwards]. A number of integral representations and infinite integrals containing functions of the hypergeometric type are obtained. A characteristic example is

$$\int_0^\infty e^{-x^2/(4z)}x^{\nu+1}[I_\nu(x)-L_\nu(x)]dx=(2z)^{\nu+1}e^z[1-\Phi(z^{\frac{1}{2}})],$$

$$\Re(z)>0, \Re(\nu)>-1,$$

where $I_\nu(x)$ is the modified Bessel function of the first kind, $L_\nu(x)$ essentially Struve's function and Φ the error function.

A. Erdélyi (Edinburgh).

Meijer, C. S. Ueber Besselsche, Struvesche und Lommelsche Funktionen. (Erste Mitteilung.) Nederl. Akad. Wetensch., Proc. **43**, 198–210 (1940). [MF 1592]

First, author proves two integral representations of the function $G_{pq}^{mn}(z)$ defined by him in an earlier paper [Nieuw Arch. Wiskde (2) **18**, part 4, 10–39 (1936)]. The simpler of the two representations runs

$$G_{pq}^{mn}\left(\xi\left|\begin{matrix}a_1, \cdots, a_p\\ b_1, \cdots, b_q\end{matrix}\right.\right)$$

$$=\frac{1}{\Gamma(\alpha-b_1)}\int_1^\infty G_{pq}^{mn}\left(\xi v\left|\begin{matrix}a_1, \cdots, a_p\\ \alpha, b_2, \cdots, b_q\end{matrix}\right.\right)(v-1)^{\alpha-b_1-1}v^{-\alpha}dv,$$

and is valid under some restrictions on variable and parameters. The other representation follows from this one by (fractional) integration by parts. In the second section there is a list of functions which can be represented by particular cases of G_{pq}^{mn}. In the third section there is a number of integral representations and integral formulae for Bessel functions which are particular cases of the two main formulae for G_{pq}^{mn}. *A. Erdélyi* (Edinburgh).

Giulotto, Virgilio. Funzioni ultracilindriche. Ist. Lombardo, Rend. **72**, 58–72 (1939). [MF 942]

The author starts from the differential equation (said to be the characteristic of hyperspherical functions)

$$(1-x^2)\frac{d^2z}{dx^2}-(2\nu+1)x\frac{dz}{dx}+\left[n(n+2\nu)-\frac{m(m+2\nu-1)}{1-x^2}\right]z=0,$$

and puts $x=\cos\theta$, yielding after some transformations the differential equation:

$$\frac{d^2z}{d\theta^2}+2\nu\frac{1}{\theta}\frac{dz}{d\theta}+\left[1-\frac{m(m+2\nu-1)}{\theta^2}\right]z=0.$$

This equation is solved by a power series solution, the coefficients of which are determined by simple recursion formulas. Introducing the gamma function, simplified expressions are derived. These power series solutions are termed hypercylindrical functions, the ordinary Bessel functions of the first kind resulting for $\nu=\frac{1}{2}$. From the power series expressions are derived for the differential quotient with respect to the independent variable, as well as for the second differential quotient. Finally, definite integral expressions with trigonometrical functions as integrands, analogous to those for Bessel functions of the first kind, are obtained. Several special cases of the formulas are discussed.

M. J. O. Strutt (Eindhoven).

Simonart, Fernand. Sur les séries hypergéométriques. Ann. Soc. Sci. Bruxelles. Sér. I. **59**, 358–375 (1939). [MF 1205]

The generalized hypergeometric series $\sum_{n=0}^{\infty}a_nx^n$ of order m studied by the author is defined by the recurrence relation

$$\frac{a_{n+1}}{a_n}=\frac{n^m+\alpha_1n^{m-1}+\cdots+\alpha_m}{(n+1)(n^{m-1}+\beta_1n^{m-2}+\cdots+\beta_{m-1})},\quad a_0=1,$$

where α, β are arbitrary constants. It is shown that this series is a particular integral, valid near $x=0$, of the linear homogeneous differential equation of order m

$$\sum_{k=0}^{m-1}x^{m-k-1}(\mu_k-\lambda_kx)\frac{d^{m-k}y}{dx^{m-k}}-\lambda_my=0,\quad \mu_0=\lambda_0=1,$$

where λ, μ are linear functions of α, β (the coefficients are Stirling numbers of the second kind). In the latter part of the paper the research is restricted to the case $m=2$. The classical results of Gauss, Kummer and Goursat are reproduced. *S. C. van Veen* (Dordrecht).

Smith, F. C. Relations among the fundamental solutions of the generalized hypergeometric equation when $p=q+1$. II. Logarithmic cases. Bull. Amer. Math. Soc. 45, 927–935 (1939). [MF 785]

Die verallgemeinerte hypergeometrische Funktion ${}_{q+1}F_q(a_1, \cdots, a_{q+1}; c_1, \cdots, c_q; z)$ genügt bekanntlich einer linearen homogenen Differentialgleichung $(q+1)$-ter Ordnung, deren Fundamentallösungen in der Umgebung der ausserwesentlichen singulären Stelle $z=0$ im allgemeinen hypergeometrische Funktionen der Gestalt ${}_{q+1}F_q(z)z^\lambda$ sind $(\lambda=0, 1-c_1, 1-c_2, \cdots, 1-c_q)$. Das Fundamentalsystem in der Umgebung der ausserwesentlichen singulären Stelle $z=\infty$ besteht im allgemeinen aus hypergeometrischen Funktionen der Gestalt ${}_{q+1}F_q(1/z)z^\mu$ $(\mu=-a_1, -a_2, \cdots, -a_{q+1})$. Ist $c_j-c_k=0, 1, 2, \cdots$, so sind gewisse Fundamentallösungen für die Umgebung des singulären Punktes $z=0$ keine hypergeometrische Reihen, sondern sehr komplizierte Funktionen von z und $\log z$. Ist $a_j-a_k=0, 1, 2, \cdots$, so gilt etwas Ähnliches für die Fundamentallösungen in der Umgebung von $z=\infty$. Verfasser leitet für diese logarithmischen Fälle die Übergangssubstitutionen zwischen den Fundamentallösungen bei $z=0$ und $z=\infty$ ab. *C. S. Meijer* (Groningen).

Rice, S. O. Some properties of ${}_3F_2(-n, n+1, \zeta; 1, p; v)$. Duke Math. J. 6, 108–119 (1940). [MF 1547]

This function is studied in the case when $n+1$ is a positive integer and $-p$ is not. Integral representations are first obtained. Three of these involve the Legendre polynomial and one involves gamma functions. Some integrals are evaluated in which the function under consideration appears under the integral sign. They give new expressions for the Legendre polynomial. Two generating functions are then given. One involves an ordinary hypergeometric function, the other a generalized hypergeometric function of type ${}_2F_2$. When ζ is a negative integer, there is a generating function expressed by means of the Legendre polynomial. The function appears also as a coefficient in a series of zonal harmonics of the second kind, a series of zonal harmonics and a series of Poisson's Bessel functions. For large values of n an asymptotic formula is obtained by an application of Darboux's method when v is of the form $\sin^2 \frac{1}{2}\phi$. This method is used also to obtain asymptotic expressions for the polynomials $F_n(x)$, $F_n(iy)$, where x and y are real. In the second case there is one factor inversely proportional to n and a trigonometrical factor with argument $2\alpha - y \log n$, where $\alpha = \arg \Gamma(\frac{1}{2}+\frac{1}{2}iy)$. *H. Bateman.*

Differential Equations

✱Droste, H. W. Die Lösung angewandter Differentialgleichungen mittels Laplacescher Transformation. Neuere Rechenverfahren der Technik, vol. 1. E. S. Mittler & Sohn, Berlin, 1939. 35 pp.

The first half of this pamphlet is devoted to those operational properties of the Laplace transformation which are most important to the theoretical engineer. Simple derivations are given and the properties are clearly displayed and well illustrated. The identification of the formal results with the rules of Heaviside's operational calculus is kept in view. The second half illustrates the use of the transformation in solving some of the ordinary and partial differential equations which appear in communication engineering. An excellent account is given of the solution of Bessel's equation by this method, for the functions $J_n(x)$, and the derivations of many properties of these functions with the aid of their transforms. The author has included references to earlier or original treatments of the topics, and a table of the transformations derived here, mostly involving Bessel functions. *R. V. Churchill* (Ann Arbor, Mich.).

Miller, F. H. A note on Heaviside's expansion theorem. J. Appl. Phys. 11, 343–346 (1940). [MF 1933]

Formal derivation of variations of the Cauchy-Heaviside expansion theorem used in solving linear constant-coefficient differential equations, for the special case of rational functions with real first-order poles. No reference is made to the equally practical and more general tables of Laplace transforms available for solution of the same types of equations. *J. L. Barnes* (Merchantville, N. J.).

Rainville, Earl D. Linear differential invariance under an operator related to the Laplace transformation. Amer. J. Math. 62, 391–405 (1940). [MF 1773]

The operator σ is defined as one which transforms linear differential operators with polynomial coefficients into operators of the same character in the following manner:

$$\sigma[\sum_{k,n} a_{kn}x^kD^n] = \sum_{k,n} a_{kn}(-1)^kD^kx^n,$$

where k and n are nonnegative integers, a_{kn} are any constants and D^kx^n is the operator which, when applied to a function $F(x)$, gives $d^k/dx^k(x^nF)$. The σ operation gives the effect produced, upon the differential operators only, by applying the Laplace integral transformation to corresponding linear differential forms. The paper is devoted to a thorough study of the differential operators which remain invariant or pseudo-invariant under σ. It is shown that every invariant can be expressed as a polynomial in the two invariants D^2+x^2 and x^2D^2+2xD. A second classification of invariants is given, together with representation theorems. Interesting applications of the theory are made to the solution of linear operational equations in σ, and to the determination of all linear differential equations invariant under σ. *R. V. Churchill* (Ann Arbor, Mich.).

Viola, Tullio. Sulla stabilità degli integrali delle equazioni differenziali lineari, omogenee e a coefficienti costanti. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 1, 238–244 (1940). [MF 1758]

The author gives a necessary and sufficient condition that all the solutions of a linear homogeneous differential equation with constant coefficients $c_0y^{(h)}+\cdots+c_{h-1}y'+c_hy=0$ be stable as $t\to\infty$. The case in which the coefficients c_i are polynomials in a set of real parameters is also considered. *W. T. Reid* (Chicago, Ill.).

Sansone, Giovanni. Valutazione dell'errore nel calcolo effettivo del periodo del moto perturbato in un caso tipico di prima approssimazione. Boll. Un. Mat. Ital. (2) 1, 422–426 (1939). [MF 1311]

Se $F(s)$ è una funzione non negativa finita e continua insieme con la $F'(s)$ nell'intervallo (s_1, s_2) e se $F(s_1)=F(s_2)=0$, $F'(s_1)\neq 0$, $F'(s_2)\neq 0$, ogni integrale dell'equazione $(ds/dt)^2$

$=F(s)$ che per $t=t_0$ assume un valore $s(t_0)$ contenuto in (s_1, s_2) è, per un teorema di Weierstrass, periodico con periodo

$$T=2\int_{s_1}^{s_2}\frac{ds}{(F(s))^{\frac{1}{2}}}.$$

Se è $F(s)=\omega^2(s-s_1)(s_2-s)+\epsilon g(s)$, con $g(s)$, $g'(s)$, $g''(s)$ continue in un intervallo contenente nel suo interno (s_1, s_2), l'autore, precisando una formula approssimata di T. Levi-Cività, dimostra che si ha:

$$T=\frac{2\pi}{\omega}-\frac{\epsilon}{\omega}\int_0^{2\pi}\lambda(s)d\alpha+\epsilon^{\frac{3}{2}}O(1),$$

con:

$$\omega^3\lambda(s)\begin{vmatrix} s^2 & s & 1\\ s_1^2 & s_1 & 1\\ s_2^2 & s_2 & 1\end{vmatrix}=\begin{vmatrix} g(s) & s & 1\\ g(s_1) & s_1 & 1\\ g(s_2) & s_2 & 1\end{vmatrix},$$

$$2s=(s_1+s_2)-(s_1-s_2)\cos\alpha.$$

C. Miranda (Torino).

Zwirner, Giuseppe. Problemi di valori ai limiti per equazioni differenziali ordinarie. Rend. Sem. Mat. Univ. Padova **10**, 35–45 (1939). [MF 1746]

The differential system $y'''=f(x, y, y', y'')$, $y'(a)=\alpha$, $y(x_0)=y_0$, $y'(b)=\beta$, $a<x_0<b$, is shown to have at least one solution under the following hypotheses: f is real-valued and continuous in its arguments on S: $a\leqq x\leqq b$, $|y|<+\infty$, $|y'|<+\infty$, $|y''|<+\infty$; $f=f_1(x, y')+f_2(x, y, y', y'')$, where f_1 is a continuous function of its arguments which is non-decreasing in y', and f_2 is uniformly bounded in absolute value on S. An analogue of this theorem for higher order equations is also given. *W. T. Reid* (Chicago, Ill.).

Zwirner, Giuseppe. Un criterio di esistenza per un problema di valori al contorno per equazioni differenziali del secondo ordine. Rend. Sem. Mat. Univ. Padova **10**, 55–64 (1939). [MF 1748]

The author extends results previously obtained [Rend. Sem. Mat. Roma **3**, 57–70 (1939); these Rev. **1**, 75 (1940)] for a differential system of the form $y''=f(x, y, y')$, $y(a)=\alpha$, $y(b)=\beta$, where f is defined in C: $a\leqq x\leqq b$, $\sigma(x)\leqq y\leqq\tau(x)$, $-\infty<y'<+\infty$. It is shown that this system admits a solution whenever: (I) f is finite, continuous in (y, y'), and measurable in x on C; (II) on arbitrary bounded subsets of C the function $f(x, y, y')$ is bounded in absolute value by a corresponding function $\psi(x)$ which is integrable on ab; (III) $|f(x, y, y')|\leqq\bar{\phi}(y)\phi(y')+\chi(x)$, where $\bar{\phi}(u)\geqq 0$ and integrable on every finite interval, $\chi(x)\geqq 0$ and integrable on ab, $\phi(u)$ is continuous and positive on $-\infty<u<+\infty$; moreover, $u/\phi(u)$ is bounded in absolute value, and the integral of this function on each of the intervals $0\leqq u<+\infty$, $-\infty<u\leqq 0$ is divergent; (IV) $\sigma(x)$ and $\tau(x)$ are of class C' on ab; (V) the functions

$$\sigma'(x)-\int_a^x f(u, \sigma(u), \sigma'(u))du,$$

$$\int_a^x f(u, \tau(u), \tau'(u))du-\tau'(x)$$

are non-decreasing on ab. Finally, it is shown that the condition that $u/\phi(u)$ be bounded in absolute value is superfluous whenever $\chi(x)\equiv 0$ on ab, and f is bounded in absolute value on arbitrary bounded subsets of C. *W. T. Reid.*

Scorza Dragoni, G. A proposito di un teorema sulle equazioni differenziali ordinarie. Rend. Sem. Mat. Univ. Padova **10**, 90–100 (1939). [MF 1739]

This paper is concerned with the same differential system as that of the paper of Zwirner reviewed above. Using the notation of that review, it is shown under hypotheses (I)–(V) that the condition that $u/\phi(u)$ be bounded in absolute value is superfluous whenever simply $\chi(x)\equiv 0$ on ab.

W. T. Reid (Chicago, Ill.).

Zwirner, Giuseppe. Sopra un teorema sulle equazioni differenziali del secondo ordine. Rend. Sem. Mat. Univ. Padova **10**, 65–68 (1939). [MF 1736]

The author presents a simplified proof of a result on differential systems which he had previously established [Un' osservazione su un problema ai limiti per le equazioni differenziali, Boll. Un. Mat. Ital. (2), 334–336 (1939)].

W. T. Reid (Chicago, Ill.).

Kamke, E. Zum Entwicklungssatz bei polaren Eigenwertaufgaben. Math. Z. **45**, 706–718 (1939). [MF 1408]

A discussion of the ordinary differential polar boundary value problem of order n:

$$L(y)=\lambda g(x)y, \tag{1}$$

$$U_\mu(y)=0, \qquad \mu=1, 2, \cdots, n, \tag{2}$$

in which (i) $g(x)$ is continuous and changes sign a finite number of times on the fundamental interval (a, b); (ii) $L(y)$ is a self-adjoint differential expression

$$\sum_{\nu=0}^{n} f_\nu(x)y^{(\nu)}(x), \qquad f_n(x)\neq 0;$$

(iii) $$U_\mu(y)=\sum_{j=0}^{n-1}[\alpha_\mu^{(j)}y^{(j)}(a)+\beta_\mu^{(j)}y^{(j)}(b)];$$

(iv) $\int_a^b yL(y)dx\geqq 0$ for every continuous function y satisfying (2). It is shown that there exists an infinity of characteristic values λ_k, all real and of finite index, with characteristic solutions $\Psi_k(x)$; that the series

$$\Sigma\frac{1}{|\lambda_k|},\quad \Sigma\frac{\Psi_k^2(x)}{|\lambda_k|}$$

are convergent; and that any function $F(x)$ which satisfies (2) and has a continuous nth derivative is expansible in an absolutely and uniformly convergent series of characteristic solutions. The method consists in showing that the differential system is representable as a polar integral equation, in consequence of which the familiar facts of the Hilbert theory are made applicable. *R. E. Langer.*

Kamke, E. Weinsteins Einschliessungssatz. Math. Z. **45**, 788–790 (1939). [MF 1415]

A new proof, under less restrictive conditions, of Weinstein's theorem [Proc. Nat. Acad. Sci. U. S. A. **20**, 529–532 (1934)]: Let $L(y)=\lambda g(x)y$; $U_\mu(y)=0$ ($\mu=1, \cdots, n$) be a self-adjoint definite boundary value problem and let $g(x)$ be continuous and positive; let $u(x)$ be an arbitrary function, not identically zero in $a\leqq x\leqq b$, possessing n continuous derivatives and satisfying the boundary conditions; put

$$\alpha=\frac{\int_a^b uL(u)dx}{\int_a^b gu^2dx}\quad\text{and}\quad \beta^2=\frac{\int_a^b \{L(u)\}^2/g\,dx}{\int_a^b gu^2dx}.$$

Then $\beta^2 \geqq \alpha^2$, and between $\alpha-(\beta^2-\alpha^2)^{\frac{1}{2}}$ and $\alpha+(\beta^2-\alpha^2)^{\frac{1}{2}}$ there is at least one eigenvalue of the boundary value problem. *A. Erdélyi* (Edinburgh).

Rosenblatt, Alfred. Sur les points singuliers des équations différentielles. Actas Acad. Ci. Lima **2**, 59–70 (1939). [MF 1797]

This paper gives the details of a proof that the system of differential equations

$$(\Sigma A y_1^{\alpha_1} y_2^{\alpha_2} x^{\beta})\frac{dy_1}{dx} = \Sigma A^1 y_1^{\alpha_1^1} y_2^{\alpha_2^1} x^{\beta^1},$$

$$(\Sigma A y_1^{\alpha_1} y_2^{\alpha_2} x^{\beta})\frac{dy_2}{dx} = \Sigma A^2 y_1^{\alpha_1^2} y_2^{\alpha_2^2} x^{\beta^2},$$

has integrals of the form

$$y_1 = \nu_1 x^{\mu_1} e^{-k_1(\log 1/x)^m}, \quad y_2 = \nu_2 x^{\mu_2} e^{-k_2(\log 1/x)^m},$$

where $0<m<1$, $\mu_1>0$, $\mu_2>0$, and where ν_1, ν_2 approach definite, non-zero limits when x approaches zero along a path which lies within a certain positive angle α. Certain transformations built around a polyhedron of Puiseux are used in the demonstration. An announcement of the results of this paper appeared in C. R. Acad. Sci. Paris **209**, 10–11 (1939); these Rev. **1**, 16 (1940). *W. M. Whyburn.*

Petrowsky, I. G. Sur l'analyticité des solutions des systèmes d'équations différentielles. Rec. Math. N.S. [Mat. Sbornik] **5** (47), 3–70 (1939). (French. Russian summary) [MF 1424]

The principal results of this paper are concerned with elliptic systems of partial differential equations $F_i=0$ $(i=1, \cdots, N)$ involving independent variables $x_0, x_1, \cdots, x_n$ and N dependent functions $u_1, u_2, \cdots, u_N$; the system is supposed to be in normal form. If the functions F_i are analytic in all their arguments in a complex region G there exists a number m, determined in terms of n and the order of the system in the u_j, such that all the solutions of this system which are of class $C^{(m)}$ in G are also analytic in this region. In case the F_i are assumed to be analytic in the u_j and their derivatives, and in only one of the independent variables x_0, but to be differentiable in the other variables to a sufficiently high order, then the author establishes a corresponding theorem on the analytic character of a solution as a function of x_0 alone. For systems of the above form which are not elliptic the author gives conditions which insure that the system admits of non-analytic solutions.

W. T. Reid (Chicago, Ill.).

Bulgakov, B. V. Sur le mouvement troublé par des forces de haute fréquence. Compositio Math. **7**, 390–427 (1940). [MF 1872]

Consider a system of differential equations of the form

$$\frac{dx_i}{dt} = f_i(x_1, \cdots, x_n, t, \theta, \mu), \qquad i=1, \cdots, n,$$

with

$$f_i(x_1, \cdots, x_n, t, \theta, \mu) = F_i(x_1, \cdots, x_n, t, \mu) + \sum_{m=1}^{\infty} [f_{im}(x_1, \cdots, x_n, t, \mu) \cos m\theta + \varphi_{im}(x_1, \cdots, x_n, t, \mu) \sin m\theta],$$

and $\theta=[(t-t_0)/\mu]+\theta_0$, where μ is a small parameter. The functions F_{im}, f_{im} and φ_{im} are supposed continuous together with their partial derivatives up to a certain order, and the series $\sum_m |f_{im}|$ and $\sum_m |\varphi_{im}|$ are supposed to converge when the point $(x_1, \cdots, x_m, t, \mu)$ is in the domain considered. In parts of the paper still further hypotheses are used. The paper is largely concerned with the construction and convergence of approximate solutions. The interval $t_0 \leqq t \leqq T$, on which the solution is desired, is split into sub-intervals $t_{i-1} \leqq t \leqq t_i$, $i=1, \cdots, \nu$, with $t_\nu = T$. The solution on the sub-interval $t_0 \leqq t \leqq t_1$ is obtained by the method of successive approximations, using for the first approximation the set of constants afforded by a given set of initial conditions for the differential equations. The values of a high approximation at $t=t_1$ are then used for initial values on the interval $t_1 \leqq t \leqq t_2$, the method of successive approximations being again applied. This procedure is repeated until an approximation is obtained for the whole interval $t_0 \leqq t \leqq T$.

Uniform convergence is obtained in two ways. Either we may fix the subdivision and continue the process of successive approximation in each sub-interval further and further; or we may fix the number of steps in the successive approximation for all the sub-intervals and let $\nu \to \infty$ and $\max |t_i - t_{i-1}| \to 0$. The method is thus a combination of the Cauchy-Lipschitz method and the Picard method of successive approximations. *D. C. Lewis* (Durham, N. H.).

Saltykow, N. Invariants des équations linéaires aux dérivées partielles du second ordre. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. **5**, 109–119 (1939). [MF 1917]

It is the purpose of this paper to show that certain quantities which have previously been given by a recurrence formula and which are related to the integration in finite terms of the linear equation of the second order in a single unknown and two independent variables appear directly as invariants of this equation with respect to transformations of the variables. The calculations are carried out in detail without benefit of the simplifying procedures of the tensor analysis. Several special equations are considered as examples. *T. Y. Thomas* (Los Angeles, Calif.).

Picone, Mauro. Nuovi metodi per il calcolo delle soluzioni delle equazioni a derivate parziali della fisica matematica. Ann. Sci. Univ. Jassy **26**, 183–232 (1940). [MF 1658]

The author shows how linear boundary value problems in the partial differential equations of physics can be resolved into an infinite system of integral equations $(F_s, U)=c_s$ $(s=1, 2, \cdots)$ of the Fischer-Riesz type. The given vectors F_s and the unknown one U have as components functions defined over different spaces, such as the interior of the region of the boundary problem and parts of the boundary. The inner product here represents the sum of the usual inner products of the corresponding components of F_s and U, and the c_s are known constants. The solution of this system is given in terms of that of an infinite system of algebraic equations; thus a method of obtaining approximate solutions is obtained by using only a finite number of the algebraic equations. The method is illustrated by a few problems in elasticity, for which approximate polynomial solutions are found. The procedure is lengthy and involved. According to the examples treated it would seem that the choice of any other than a polynomial form of the approximate solution would in general so increase the complexity of the process as to make it quite impractical.

R. V. Churchill (Ann Arbor, Mich.).

Eichler, M. Allgemeine Integration einiger partieller Differentialgleichungen der mathematischen Physik durch Quaternionenfunktionen. Comment. Math. Helv. **12**, 212–224 (1940). [MF 1570]

R. Fueter developed in recent years the theory of functions of quaternion variables. This theory can be applied to several boundary problems of (classical) mathematical physics. First, author attacks the problem of an electromagnetic field. Here he obtains the formula which has been derived in a different way by Gr. C. Moisil and N. Théodoresco [Mathematica, Cluj **5**, 142 (1931)]. He specializes this solution for static fields. Quaternion functions which represent an electrostatic field are called static functions. From his results he derives the Poisson formula for the three-dimensional sphere. The second part of the paper deals with the power series expansions of the theory of functions of quaternions. The polynomials which occur in the terms of this expansion are defined in a way slightly different from Fueter's definition. Written as symbolical powers, their differentiation goes according to the well-known rules for ordinary powers. Also there is a generalization of the binomial theorem, as well as an exponential defined by the usual expansion (with symbolical powers which are Fueter polynomials). The theory of functions of quaternions and of the expansion into power series of such functions can be used in deriving general theorems on solutions of linear partial differential equations of second order in three variables with constant coefficients. The power series expansions give the expansion of a potential function in a series of harmonic functions. *A. Erdélyi* (Edinburgh).

Southwell, R. V. On the natural frequencies of vibrating systems. Proc. Roy. Soc. London. Ser. A. **174**, 433–457 (1940). [MF 1653]

Following principles from Lord Rayleigh's "Theory of Sound" and, as the reviewer may remark, disregarding some more recent literature, the author discusses a method of finding a lower bound for eigenvalues (natural frequencies) of vibrating systems, in particular for the lowest frequency and for systems with a finite degree of freedom. The method is exemplified by numerical discussion of the clamped bar and circular disc. The second part analyzes an alleged paradox, consisting in the fact that the natural frequencies of the free bar and the clamped bar are seemingly identical, whereas, according to the theory, the latter ought to be higher than the former. The author could have dismissed this paradox with the remark that the two first eigenvalues of the free bar, which have to be counted, are zero. They belong to rigid displacements as "modes" of motion. *R. Courant* (New York, N. Y.).

Weinstein, Alexandre. Sur la théorie unitaire des valeurs propres des membranes et des plaques encastrées. C. R. Acad. Sci. Paris **210**, 161–163 (1940). [MF 1630]

The author gives a procedure for finding the eigenvalues for the equation for the vibration of a plate in the shape of a domain S, the plate being clamped at the edges, when the eigenvalues and eigenfunctions for the equation of a vibrating membrane on S are known. Suppose that $\omega_1<\omega_2<\cdots$ are the distinct eigenvalues on S for the membrane, that r_n is the multiplicity of ω_n, that $u_1^{(n)}, \cdots, u_{r_n}^{(n)}$ are for each n a complete orthonormal set of eigenfunctions corresponding to ω_n, and that $p_1, p_2, \cdots$ are a complete set of linearly independent harmonic functions on S. Then, in general, the eigenvalues for the plate are roughly speaking the roots of the ∞^2 determinant $|M|=0$, M being the matrix $a_{ik}(\rho)$, where

$$a_{ik}(\rho)=\sum_{n=1}^{\infty}\sum_{\nu=1}^{r_n}(p_i, u_\nu^{(n)})(p_k, u_\nu^{(n)})\cdot\frac{\omega_n^2}{\omega_n^2-\rho},$$

(f, g) denoting the ordinary inner product of two functions on S. If S is such that the sequence $p_1, p_2, \cdots$ breaks up into sets $p_1^{(1)}, \cdots, p_{r_1}^{(1)}; p_1^{(2)}, \cdots, p_{r_2}^{(2)}; p_1^{(3)}, \cdots, p_{r_3}^{(3)}; \cdots$ of $r_1, r_2, \cdots$ functions, each such that $(p_i^k, u_\nu^{(n)})=0$ unless $k=j_n$, where $r_{j_n}=r_n$, in which case the determinant $|(p_i^{j_n}, u_k^{(n)})|\neq 0$, $i, k=1, \cdots, r_n$, then the matrix M breaks up into finite matrices $M_1, M_2, \cdots$ of order $r_1, r_2, \cdots$ arranged along the principal diagonal. In this case the eigenvalues for the plate are simply the roots of the determinants $|M_n|=0$, $n=1, 2, \cdots$.

Using the above result, the author obtains the eigenvalues for a clamped plate for a circle, an annular ring and a rectangle. The author also treats certain mixed problems and the problem of the buckling of a plate for the same regions. *C. B. Morrey* (Berkeley, Calif.).

Soboleff, S. Sur la théorie des équations hyperboliques aux dérivées partielles. Rec. Math. N.S. [Mat. Sbornik] **5** (47), 71–99 (1939). (Russian. French summary) [MF 1425]

The author investigates the existence of the solution of a hyperbolic quasilinear solution in n independent variables in the case of the space-like initial problem. He shows that, as in the linear case, it suffices to assume quadratic integrability of the initial values and their derivatives up to the order $[n/2]+3$ to obtain this same property for the solution on any surface. His result implies a similar result for the general nonlinear equation, if $[n/2]+3$ is replaced by $[n/2]+4$. The tools are estimates for certain "energy" integrals in the more complicated case where the coefficients in these integrals depend themselves on the functions and derivatives to be bounded by the estimates.
H. Lewy (Berkeley, Calif.).

Mangeron, Demetrio. Sulle equazioni lineari a derivate parziali di tipo "composito" secondo Hadamard. Boll. Un. Mat. Ital. (2) **2**, 18–25 (1939). [MF 1307]

L'unicità di una soluzione dell'equazione

$$\frac{\partial^4 u}{\partial x^4}-\frac{\partial^4 u}{\partial y^4}=0$$

verificante le condizioni di Cauchy:

$$\left(\frac{\partial^i u}{\partial y^i}\right)_{y=0}=S_i(x), \quad a\leqq x\leqq b;\, i=0, 1, 2, 3,$$

é assicurata nel quadrato Q definito dalle limitazioni $a\leqq x\leqq b$, $|y|\leqq x-a$, $|y|\leqq b-x$. Se inoltre, essendo S_0 e S_1 dotate rispettivamente di derivate quinta e quarta a variazione limitata in un intervallo (α, β), contenente (a, b) nel suo interno, e risultando $S_0, S_0'', S_0^{(4)}, S_1, S_1''$ nulle per $x=\alpha$ e $x=\beta$, si ha:

$$S_2=S_0''+M\sum_{k=1}^{\infty}\frac{k^p}{\cosh k\rho c}\sin k\rho(x-\alpha),$$

$$S_3=S_1''+N\sum_{k=1}^{\infty}\frac{k^q}{\sinh k\rho c}\sin k\rho(x-\beta),$$

con M, N, p, q, c costanti e $|c|<(b-a)/2$, $\rho=\pi/(\beta-\alpha)$, l'esistenza della $u(x, y)$ resta assicurata almeno nella parte di Q in cui si ha $|y|<|c|$. Si noti che le ipotesi indicate non implicano l'analiticità delle $S_i(x)$. Per pervenire al teorema enunciato e ad altri consimili l'autore si vale di un metodo di M. Picone [Rend. Sem. Mat. Roma, 1937], che é anche applicabile allo studio del problema di Cauchy per l'equazione $D(E(u))=f(x, y)$ con:

$$D=\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+\beta(y)\frac{\partial}{\partial y}+\gamma(y),$$

$$E=a_{20}(y)\frac{\partial^2}{\partial x^2}+a_{02}(y)\frac{\partial^2}{\partial y^2}+b(y)\frac{\partial}{\partial y}+a(y), \quad a_{20}a_{02}>0.$$

C. Miranda (Torino).

Lagrange, René. Les familles de surfaces de révolution qui possèdent des harmoniques. Acta Math. **71**, 283–315 (1939). [MF 767]

The author considers two one-parameter families of surfaces of revolution $\eta(x, y, z)=\text{const.}$ and $\theta(x, y, z)=\text{const.}$, orthogonal to each other. Expressing the differential equation of potential in the independent variables η, θ and φ (angle round the axis of revolution), he considers solutions of the form $f(\eta\theta)N(\eta)T(\theta)\Phi(\varphi)$, where Φ is a sine or cosine or exponential function. The necessary and sufficient condition for such solutions leads to the differential equation $\partial^2(\Delta \lg \rho)/\partial\eta\partial\theta=0$, where ρ is the distance from the axis of revolution. Inversion of these solutions with respect to a point on the axis of revolution leads to a series of harmonic solutions, identical with the first ones, with a common multiplier. The discussion of the above fundamental differential equation, taking $z+i\rho=u$ and $z-i\rho=v$, leads to two elliptic differential equations for u and v. By taking suitable values for the constants in these equations the well-known cases of cylindrical harmonics, parabolic functions, conical functions of revolution, and harmonics of elliptical and hyperbolic surfaces of revolution are obtained. A new set of surfaces conforming to the above conditions is found, consisting of cycloids of revolution. As special cases of these new general solutions, spherical harmonics and toroidal harmonics result, as well as families of bicircular quartics, homofocal to the axis of revolution. These latter solutions are studied in some detail. *M. J. O. Strutt* (Eindhoven).

Monna, A. F. Sur la capacité des ensembles. Nederl. Akad. Wetensch., Proc. **43**, 81–86 (1940). [MF 1236]

En considérant pour un ensemble borné quelconque les ensembles ouverts contenants et les ensembles fermés contenus, l'auteur introduit un potentiel capacitaire extérieur et un autre intérieur, ce dernier, par exemple, étant l'enveloppe supérieure des potentiels capacitaires des ensembles fermés contenus. Il introduit de même les capacités extérieure et intérieure. Propriétés diverses; en particulier la mesure extérieure tend vers zéro avec la capacité extérieure. *M. Brelot* (Bordeaux).

Monna, A. F. Sur les notions de "point stable" et "point régulier" dans le problème de Dirichlet. Nederl. Akad. Wetensch., Proc. **43**, 87–90 (1940). [MF 1237]

Poursuivant des recherches antérieures [these Rev. **1**, 122 (1940)] sur l'extension des notions de point-frontière régulier ou stable, dans le cas d'un ensemble E non plus ouvert ou fermé mais borélien, l'auteur généralise ici divers critères connus dans le cas de E ouvert ou fermé. *M. Brelot.*

Brelot, Marcel. Critères de régularité et de stabilité. Acad. Roy. Belgique. Bull. Cl. Sci. **25**, 125–137 (1939). [MF 1301]

The author discusses criteria for regularity and stability with respect to harmonic functions. The measure distribution and measure potential of a set are defined as those pertaining to the mass distribution which is the measure function of the set. Let Ω be an open set of frontier Σ, O a point of Σ, S an open sphere of center O and v the measure potential of $\Omega\cdot S$. Let u be the sequence solution of Laplace's equation defined by the values of v on the frontier of $\Omega\cdot S$. A criterion of the regularity of O is that $u(M)\to v(O)$, when $M\to O$ from Ω. Criteria are also given for the regularity of a sequence M_n of points of Ω tending to O.

Let F be bounded and closed, with frontier F^* on which $\phi(M)$ is given as continuous. It may be extended continuously everywhere as a function $\Phi(M)$. Let Ω_n be a decreasing sequence of open sets (with boundaries regular for the Dirichlet problem) with limit F. The solution u_n of the classical problem for Ω_n and Φ tends to a function u on F independent of the extension Φ and Ω_n (being harmonic at interior points of F if there are any). A point O of F^* is stable if $u(O)=\phi(O)$, whatever ϕ is given. Criteria for stability are given similar to those for regularity. The analogue of the sweeping-out process is called extremalization; given a distribution of positive mass on F, of potential w, the extremal w^0 of w on F is the minimal function which is superharmonic in an open set containing F, agreeing with w outside F. If the open set is the space outside F, the mass distribution of w^0 is called the extremalization of the given one. The unstable frontier points of F^* are the points of discontinuity of the extremalized measure potential of F. By means of the extremalization of unit mass at a point P of F, a distribution ν^P is obtained, and the extremalization of general mass is given in terms of this by a formula corresponding to the familiar one for the sweeping-out process. The unstable points of F^* bear none of the mass ν. *G. C. Evans* (Berkeley, Calif.).

Integral Equations, Functional Analysis, Ergodic Theory

Iglisch, Rudolf. Über lineare Integralgleichungen mit vom Parameter abhängigem Kern. Math. Ann. **117**, 129–139 (1939). [MF 1388]

The object of this paper is to point out some results relative to the characteristic values of symmetric kernels of an integral equation: $z(x)-\lambda\int K(x, y; \lambda)z(y)dy=f(x)$, which are functions of the parameter λ, results suggested by C. Miranda [Rend. Circ. Mat. Palermo **60**, 286–304 (1937)], who treats symmetric kernels of the form

$$K(x, y; \lambda)=K(x, y)+\sum_{i=1}^{n}\frac{\lambda}{\lambda+a_i}H_i(x, y),$$

$H_i(x, y)$ positive of finite order. For instance, if $K(x, y; \infty)$ has an infinite number of positive characteristic values, and $K(x, y; \lambda_1)-K(x, y; \lambda_2)$ is positive definite for $\lambda_1>\lambda_2\geqq k>0$, then $K(x, y; \lambda)$ has an infinite number of characteristic values greater than k; $\lambda=\lambda_0$ is not a limiting value of characteristic values of $K(x, y; \lambda)$ if $K(x, y; \lambda)-K(x, y; \lambda_0)$ is positive definite for $\lambda>\lambda_0$ and negative definite for $\lambda<\lambda_0$.

More particularly, if

$$K(x, y; \lambda) = G(x, y) + \sum_n H_n(x, y)\frac{a_n\lambda + b_n}{c_n\lambda + d_n},$$

H_n positive definite, $a_nd_n - b_nc_n > 0$, the sum converging uniformly in every finite region excluding $\lambda = -d_n/c_n$, $K(x, y; \lambda)$ has only real characteristic values, and possesses at least one such value, if $d_n \neq 0$ and the sum is finite.

T. H. Hildebrandt (Ann Arbor, Mich.).

Godefroy, Marcel. Sur l'application d'une méthode directe au problème de L. Lichtenstein. C. R. Acad. Sci. Paris **210**, 290–291 (1940). [MF 1601]

The author had developed [ibid. **209**, 593 (1939); cf. these Rev. 1, 146 (1940)] a method of solving functional equations, using the metric notions of Fréchet's theory. Now he applies this method to a hydrodynamic problem, proposed by L. Lichtenstein: to solve 3 integro-differential equations of the type

$$\frac{\partial x(a, b, c, t)}{\partial t} = \frac{1}{2\pi}\int\int\int_T \frac{\eta'(z-z') - \zeta'(y-y')}{r^3} d\tau',$$
$$r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2,$$

where

$$\xi = \xi_0(a, b, c)\frac{\partial x}{\partial a} + \eta_0\frac{\partial x}{\partial b} + \zeta_0\frac{\partial x}{\partial c},$$

and where ξ_0, η_0, ζ_0 are given functions of a, b, c of a general character. He proves the existence of solutions which are equal to a, b, c for $t = t_0$ and fulfill Lipschitz conditions. Further, he outlines some applications to the motion of a perfect incompressible fluid. *E. D. Hellinger* (Evanston, Ill.).

Artemenko, A. La forme générale d'une fonctionnelle linéaire dans l'espace des fonctions à variation bornée. Rec. Math. N.S. [Mat. Sbornik] **6** (48), 215–220 (1939). (Russian. French summary) [MF 1354]

Let V_c be the space of functions continuous and of bounded variation over $[0, 1]$, where $\|x\|$ is defined as the total variation of x over $[0, 1]$. The construction of the general linear functional on V_c is based on the possibility of determination of a non-denumerable sequence of functions $\{x_\alpha(t)\}$ of V_c and splitting of V_c into a sum of linear spaces each of which consists of functions absolutely continuous relative to $x_\alpha(t)$. It should be observed that the problem treated by the author has been solved already by Hildebrandt [Bull. Amer. Math. Soc. **44**, 75 (1938)] by using a generalized Stieltjes integral. *J. D. Tamarkin.*

Plancherel, M. Quelques remarques sur la théorie des transformations linéaires bornées des fonctions de plusieurs variables dans les espaces fonctionnels L^α. Comment. Math. Helv. **12**, 225–232 (1940). [MF 1571]

Let L_m^α be the class of functions of m variables, measurable and such that the αth powers of their absolute values are Lebesgue integrable over the whole m-dimensional space. The author proves: If T' is a linear transformation of L_p^α into L_q^β, and T'' a linear transformation of L_{m-p}^α into L_{m-q}^β, then, in case $0 < \alpha \leqq \beta$, there exists a unique transformation T of L_m^α into L_n^β such that $T(f'f'') = T'f' \cdot T''f''$ whenever $f' \epsilon L_p^\alpha$ and $f'' \epsilon L_{m-p}^\alpha$. The bound b_T of T is equal to the product $b_{T'} b_{T''}$. Under the same conditions, for any function $f(x) = f(x', x'')$ of L_m^α, we have

$$T(f(x); y) = T''[T'(f(x', x''); y'); y'']$$
$$= T'[T''(f(x', x''); y''); y'].$$

At the end of the paper the author considers extensions to L_m^α $(1 \leqq \alpha < 2)$ of some linear transformations defined over L_m^2. *J. D. Tamarkin* (Providence, R. I.).

Price, G. Baley. The theory of integration. Trans. Amer. Math. Soc. **47**, 1–50 (1940). [MF 1068]

Certain methods which have been used by Birkhoff, Bochner, Dunford, Gowurin and Graves for the Riemann or Lebesgue integration of vector valued functions have been extended to apply to the integration of functions whose values are bounded sets in a B-space $\mathfrak{B}$. The measure function is usually taken to be one defined over a restricted Borel field and having its values in the ring of linear operators over the space $\mathfrak{B}$. Unexpected, perhaps, is that the various classes of integrable functions arising out of this generalization do not possess the inclusion relations of the original classes, for example, the generalized Birkhoff integral no longer includes the generalized Bochner integral, etc. Some applications are made to the Fourier series of functions of the type described above. *N. Dunford.*

Izumi, Shin-ichi. An abstract integral. Proc. Imp. Acad., Tokyo **16**, 21–25 (1940). [MF 1896]

X is a set of points, $\mathfrak{X}$ is an additive field of subsets of X with an additive measure function $v(E)$, $\bar{\mathfrak{X}}$ is a completely additive field with a measure function mE. A Riemann integral $\int f dv$ and a Lebesgue integral $\int f dm$ are defined by the Darboux process and successive subdivisions. A double field integral depends on $\mathfrak{X}$ and $\bar{\mathfrak{X}}$ with $E \epsilon \mathfrak{X}$ and $\bar{E} \epsilon \bar{\mathfrak{X}}$ implying $E \cdot \bar{E} \epsilon \bar{\mathfrak{X}}$. An upper function $\varphi(E, \lambda)$ is defined by the two conditions: (a) $\varphi'(E, \lambda)$ is the least upper bound of y such that $m[E \cdot E_x(f(x) < y)] \leqq \lambda v(E)$, and (b) $\varphi(E, \lambda)$ is the greatest lower bound of y such that $m[E \cdot E_x(f(x) < y)] = m[E \cdot E_x(f < \varphi')]$. A lower function $\psi(E, \mu)$ is determined similarly by reversing the inequalities and the bounds. This gives rise to two Riemann sums on $\delta = [E_1 \cdots E_n]$ of X, $\bar{M}_\lambda(\delta) = \sum \varphi(E_i, \lambda) v(E_i)$, $\underline{M}_\mu(\delta) = \sum \psi(E_i, \mu) v(E_i)$. If

$$\lim_{\lambda \to \infty} \overline{\lim_{\delta}}\, \bar{M}_\lambda(\delta) = \lim_{\mu \to 0} \underline{\lim_{\delta}}\, \underline{M}_\mu(\delta),$$

then f is said to be integrable $(\mathfrak{X}, \bar{\mathfrak{X}})$. This is a generalization of a Denjoy definition of Lebesgue integral via Riemann sums [see C. R. Acad. Sci. Paris **193**, 695 (1931)].

T. H. Hildebrandt (Ann Arbor, Mich.).

Bochner, S. Integration and differentiation in partially ordered spaces. Proc. Nat. Acad. Sci. U. S. A. **26**, 29–31 (1940). [MF 1001]

Let S be a complete partially ordered linear vector space. By definition $|a| = \sup(a, 0) - \inf(a, 0)$ and a sequential topology in S is defined by the statement $a_n \to 0$ if $\inf_n \sup(|a_n|, |a_{n+1}|, \cdots) = 0$. An axiom of Kantorovitch is assumed which makes this notion of convergence equivalent to dominated convergence, that is, for some a_0 we have $|a_n| \leqq \epsilon a_0$ for $n \geqq N(\epsilon)$. The Lebesgue integral for functions having values in S is introduced via Cauchy sequences of simple functions. The conditions that a.c. set functions are indefinite integrals is a property of the space S and not of the σ-field of measurable sets, provided no sets of positive measure are indivisible. The space S has this property if the following is assumed. There is a sequence $L_n(a)$ of numerical additive continuous positive functionals on S satisfying the conditions (i) $L_n(a) > 0$ for all n implies $a > 0$ and (ii) $a_1 \leqq a_2 \leqq \cdots$, $\lim_m L_n(a_m) < \infty$ for all n implies $\sup a_m < \infty$. This assumption also yields a strong ergodic theorem as

well as the fact that the Fourier coefficients a_n, b_n of $f(t)$ satisfy $\sum|a_n|+|b_n|<\infty$ provided f satisfies a Lipschitz condition of order greater than 1/2. *N. Dunford.*

Smiley, M. F. A note on measure functions in a lattice. Bull. Amer. Math. Soc. **46**, 239–241 (1940). [MF 1824]

The author shows that the main properties of Carathéodory's classical definitions of "measurability" with respect to an "outer measure," and of a "regular" outer measure, can be extended to functionals on arbitrary modular lattices. This is especially interesting, as even Carathéodory's recent abstract theory of "Soma" applies only to generalized Boolean algebras. *G. Birkhoff.*

Phillips, R. S. A decomposition of additive set functions. Bull. Amer. Math. Soc. **46**, 274–277 (1940). [MF 1832]

Let $\mathfrak{T}$ be a completely additive family of subsets τ of an abstract space T. By A denote the bounded additive functions on $\mathfrak{T}$ to the reals and by C the completely additive functions. Suppose that D is any one of the four classes A, C, A_p, C_p, where A_p and C_p are the nonnegative elements of A and C, respectively, and that $\overline{\overline{\tau}}$ denotes the cardinal number of τ. The author establishes the following analogue to the Lebesgue decomposition theorem. If $\aleph$ is an infinite cardinal not greater than $\overline{\overline{T}}$, then for every $\alpha\epsilon D$ there is a unique decomposition $\alpha=\alpha_1+\alpha_2$ with these properties: $\alpha_1, \alpha_2\epsilon D$, $\alpha_2(\tau)=0$ if $\tau\epsilon\mathfrak{T}$ and $\overline{\overline{\tau}}\leqq\aleph$, and $\alpha_1(\tau)=\alpha(\tau\cdot\tau_1)$ for $\tau\epsilon\mathfrak{T}$, where $\tau_1\epsilon\mathfrak{T}$ and $\overline{\overline{\tau}}_1\leqq\aleph$. A function α defined on $\mathfrak{T}$ is $\aleph$-homogeneous if $\alpha(\tau)=\alpha(\tau\cdot R_\alpha)$, where $R_\alpha\epsilon\mathfrak{T}$ and $\overline{\overline{R}}_\alpha=\aleph$, and if $\alpha(\tau)=0$ when $\overline{\overline{\tau}}<\aleph$. The author's second result is that for each $\alpha\epsilon D$ that vanishes for single points, but not identically, there is a unique decomposition $\alpha=\sum\alpha_i$, the sum being absolutely convergent, such that α_i is $\aleph_i$-homogeneous and $\aleph_i\neq\aleph_j$ for $i\neq j$. These two theorems can be extended to completely additive functions having their values in a Banach space. *B. J. Pettis* (Cambridge, Mass.).

Pauc, Chr. Étude d'une fonctionnelle généralisant la longueur de courbe dans les espaces à écart positif uniformément continu. Revue Sci. (Rev. Rose Illus.) **77**, 658–661 (1939). [MF 1679]

Dans sa théorie des courbes [Ergebn. Math. Kolloqu. (1936)] Menger étudie les qualités de la longueur λ, respectivement la longueur supérieure $\bar{\lambda}$, d'une courbe dans un espace à écart positif, symétrique et uniformément continu. L'auteur transpose cette théorie à un espace distancié H à écart positif et uniformément continu mais non plus nécessairement symétrique, en remplaçant la fonctionnelle $\bar{\lambda}$ par la borne supérieure $\bar{\lambda}^*$ des longueurs des sous-polygones d'une courbe. Il montre que les résultats principaux de Menger peuvent être maintenus ainsi et que les qualités de $\bar{\lambda}^*$ dans l'espace H sont essentiellement les mêmes que celles de $\bar{\lambda}$ dans les espaces considérés par Menger. Voici les plus importantes: $\bar{\lambda}^*$ est une fonctionnelle semi-continue inférieurement sur les courbes de H; tout système de courbes situées sur un ensemble compact de H et dont les $\bar{\lambda}^*$ sont inférieurs à un nombre fixe fini admet une sous-suite convergeant vers une courbe de H. Les derniers alinéas étendent la théorie métrique de Menger et ses théorèmes d'existence de minimantes. *P. Scherk* (New York, N. Y.).

Reid, William T. A theorem on continuous functions in abstract spaces. Bull. Amer. Math. Soc. **46**, 113–116 (1940). [MF 1256]

The author defines an "enclosure" space as a collection of elements S to each non-vacuous subset M of which corresponds a unique non-vacuous "enclosure" set $\bar{M}$ subject to the single condition that, if M_1 and M_2 are any non-vacuous sets, $\overline{(M_1+M_2)}=\bar{M}_1+\bar{M}_2$ (it is not assumed that $M\subseteqq\bar{M}$). Two sets M and N are mutually separated if $M\bar{N}+\bar{M}N=0$, and a set X is connected if it is not the sum of two non-vacuous mutually separated sets. Let $\varphi(p)$ be single valued on S, let $\varphi(M)$ be the set of (numerical) values of $\varphi(p)$ for $p\epsilon M$; $\varphi(p)$ is said to be continuous on S if, for every M of S, $\varphi(\bar{M})$ is included in the closure of $\varphi(M)$. The author then proves the fundamental result: Let S be a connected "enclosure" space and let $f(p, q)$ be a single valued real function defined for each pair (p, q) of S such that (i) $f(p, p)=0$, (ii) $f(p, q)=f(q, p)$, (iii) $f(p, q)$ is continuous in its arguments separately on S. Then either $f\geqq 0$ or $f\leqq 0$ for all (p, q) or there exists a pair (p, q) with $p\neq q$ such that $f(p, q)=0$. Now, suppose S is any space and suppose that a real function $d(p, q)$ is defined for each pair (p, q) of points of S and satisfies (a) $d(p, p)=0$, (b) $d(p, q)=d(q, p)$, (c) $d(p, q)\neq 0$ if $p\neq q$. We say that S is topologized by $d(p, q)$ if $p_n\to p$ if and only if $d(p_n, p)\to 0$; closed and connected sets are then defined as usual. Using the above theorem the author next establishes the following results: (1) if S is connected and topologized by $d(p, q)$ and if $d(p, q)$ is continuous in each argument separately on S, then either $d(p, q)>0$ or $d(p, q)<0$ for every pair (p, q) with $p\neq q$, that is, S is semi-metric with respect to $d(p, q)$ or with respect to $-d(p, q)$; (2) if $d(p, q)$ is as above and greater than 0 and if T is a continuous transformation of S into a subset of itself in which $d(Tp_1, Tq_1)>d(p_1, q_1)$ and $d(Tp_2, Tq_2)<d(p_2, q_2)$, then there exists a point pair $(p, q)(p\neq q)$ such that $d(Tp, Tq)=d(p, q)$.

The author then defines a conditionally compact transformation T as one such that the sequence $[p, Tp, T^2p, \cdots]$ is conditionally compact for each p of S. From the above theorems and as extension of a result of D. Montgomery [Bull. Amer. Math. Soc. **40**, 620–624 (1934)], the author demonstrates the final result: Let S be a semi-metric space whose distance function $d(p, q)$ is continuous in its arguments jointly. If T is a 1-1 continuous transformation of S into the whole of itself such that T and T^{-1} are conditionally compact, then there exists a pair (p, q) with $p\neq q$ such that $d(Tp, Tq)=d(p, q)$. *C. B. Morrey* (Berkeley, Calif.).

Eidelheit, M. Concerning rings of continuous functions. Ann. of Math. **41**, 391–393 (1940). [MF 1820]

If T is a compact metric space, with elements t, denote by $E(T)$ the ring of all real continuous functions $x(t)$ defined on T; $E(T)$ becomes a separable Banach space if normed by the maximum of $|x|$. A subring of T is a linear manifold of T which, with x and y, contains $x\cdot y$. The author proves the following theorem conjectured by S. Mazur: If P is a closed subring of $E(T)$ which contains the unit element, then there is a compact metric space T_1 such that P is ring-isomorphic to $E(T_1)$. *J. A. Clarkson* (Philadelphia, Pa.).

Kondô, Motokiti. Sur les opérations analytiques des fonctions. Proc. Imp. Acad., Tokyo **15**, 193–199 (1939). [MF 1137]

This paper is connected with earlier work of the author [J. Fac. Sci. Hokkaido Imp. Univ. (1) **7**, 1–34 (1938)] and of L. Kantorovitch and E. Livenson [Fund. Math. **18**, 214–279 (1932) and **20**, 54–97 (1933)]. If there is associated with a sequence $\{F_n(x)\}$ of real functions (on a metric space R)

another such function Φ such that the value of Φ at each $x_0 \epsilon R$ depends only on the values of the $F_n(x)$ there, we say that $\Phi=\Phi(\{F_n(x)\})$ is an analytical operation on functions. Such an operation is the aggregate of its local operations, which are operations on numbers, producing from the sequence $\{F_n(x_0)\}$ the number $\Phi(x_0)$. If each $F_n(x)$ is the characteristic function of a set $E_n \subset R$ and if Φ assumes only the values 0 or 1, then Φ can be regarded as an analytical operation on sets, producing from the sequence $\{E_n\}$ the set whose characteristic function is $\Phi(\{F_n(x)\})$. Several theorems concerning the structure, the representation (as sieves) and the interrelations of the various kinds of operations are given. *J. Todd* (Belfast).

Kondô, Motokiti. Sur une extension de la théorie des fonctions de Baire. Proc. Imp. Acad., Tokyo **15**, 200–206 (1939). [MF 1138]

Extensions of the methods used by Lebesgue in his study of functions representable analytically [J. Math. Pures Appl. (6) **1**, 139–216 (1905)] are applicable to the theory of analytical operations (see preceding review). In these extensions the concept of an analytical operation (on functions) saturated with respect to an analytical operation (on sets) is important. The author establishes the existence (in certain cases) of an operation saturated with respect to a given operation and indicates various global and local properties of such operations. *J. Todd* (Belfast).

Radó, Tibor and Youngs, J. W. T. On upper semi-continuous collections. Acta Litt. Sci. Szeged **9**, 239–243 (1940). [MF 1228]

The purpose of this paper is to exhibit in a simple way the equivalence in a very general space of the different notions of an "upper semi-continuous collection" of sets advanced independently by R. L. Moore, P. Alexandroff and B. de Kerékjártó. The purpose is accomplished by showing that each is an answer to a fundamental question, the answer to which is proved in this paper to be unique. The spaces considered are "L^* spaces" which are limit spaces in which the convergence satisfies the conditions:

L_1: $a, a, a, \cdots \to a$;

L_2: if $a_n \to a$, then any subsequence $a_{n_j} \to a$;

L_3: if a_n does not tend to a, there is a subsequence a_{n_j} no subsequence of which tends to a.

By a partition $\mathfrak{F}$ of a space S is meant a collection of mutually exclusive subsets of S. One of these sets is a compartment of $\mathfrak{F}$ and $\Sigma(\mathfrak{F})$ denotes the set of these compartments. Such a partition defines a unique transformation T from S to Σ and also defines a unique equivalence relation $E(\mathfrak{F})$, where $a \sim b$ if a and b belong to the same compartment. This equivalence $E(\mathfrak{F})$ is said to be compatible with the topology of S if and only if $a_n \to a$, $b_n \to b$, $a_n \sim b_n(\mathfrak{F})$ for $n=1, 2, \cdots$ imply $a \sim b(\mathfrak{F})$. The general question mentioned above is the following: Given S_1 a compact L^* space and S_2 an untopologized set of elements, and a single valued transformation $a_2 = Ta_1$ from S_1 to S_2, in how many ways can convergence be introduced into S_2 so that S_2 is an L^* space and T is continuous? It is shown (1) that there is at most one such way to do this if it is possible at all and (2) a necessary and sufficient condition that it be possible is that the equivalence $E(\mathfrak{F})$ be compatible with the topology of S_1, $\mathfrak{F}$ being the partition of S_1 into the sets $s(a_2)$ consisting of all the elements of S_1 carried by T into an element a_2 of S_2. Such a collection of sets or partition $\mathfrak{F}$ of S_1 is upper semicontinuous if and only if $E(\mathfrak{F})$ is compatible with the topology of S_1. *C. B. Morrey* (Berkeley, Calif.).

Otchan, G. Sur l'équivalence des familles d'ensembles mesurables *B*. C. R. (Doklady) Acad. Sci. URSS (N.S.) **23**, 753–755 (1939). [MF 1956]

Two families of subsets of a complete, separable, metric space are equivalent if there is a one-one transformation of the space into itself, which takes one family into the other. A family of Borel sets is an N-family if (i) every open set belongs to the Borel ring over the family, (ii) every compact perfect set contains an element of the family. (1) An N-family all of whose sets have a certain topologically invariant property cannot be equivalent to a family none of whose sets have this property. For $\alpha \geqq 2$, the family $F^\alpha[G^\alpha]$ is not equivalent to any subfamily of $G^\alpha[F^\alpha]$, and, in particular, to no family of sets of lower class. (2) A maximal (with regard to a topologically invariant property) N-family is not equivalent to any of its proper parts. The family of Borel sets is not equivalent to any of its proper parts. The family of analytic sets is not equivalent to any of its proper parts. The family of CA-sets is not equivalent to any of its proper parts.
J. W. Tukey (Princeton, N. J.).

Pospíšil, Bedřich. On bicompact spaces. Publ. Fac. Sci. Univ. Masaryk 1939, no. **270**, 16 pp. (1939). [MF 1026]

This paper furnishes evaluations of the cardinal numbers, characters, point-characters and allied invariants for an impressive variety of topological spaces, particularly for the so-called Boolean spaces [attached to Boolean rings in the manner described by the reviewer, Trans. Amer. Math. Soc. **41**, 375–481 (1937)]. The key calculations relate to the two following Boolean spaces: (1) the space representing the ring of all subsets of a fixed class of infinite cardinal number $\mathfrak{h}$; (2) the space representing the homomorphic ring obtained therefrom by reduction modulo the ideal of all subsets of cardinal number less than $\mathfrak{h}$. If exp $\mathfrak{x}$ be written for $2^{\mathfrak{x}}$, the calculations show that both spaces have exp exp $\mathfrak{h}$ points (corresponding to the prime ideals in the associated rings), among them exp exp $\mathfrak{h}$ points of the greatest possible character, exp $\mathfrak{h}$. Both spaces, though normal, are shown not to be completely normal. These facts permit the discussion of many important Boolean rings and spaces arising in topology or in measure theory: for example, the ring of all Lebesgue-measurable sets in the plane has exp exp $\aleph_0$ elements, exp exp exp $\aleph_0$ ideals, and exp exp exp $\aleph_0$ prime ideals of character exp exp $\aleph_0$; and the ring obtained therefrom by reduction modulo the ideal of all sets of zero measure has exp $\aleph_0$ elements, exp exp $\aleph_0$ ideals and exp exp $\aleph_0$ of character exp $\aleph_0$. The full list of such results cannot be reproduced here. Suffice it to say that these results go beyond those already available in the literature [Pospíšil, Ann. of Math. **38** (1937); Tarski, Fund. Math. **16**, 181–304 (1930), and **32**, 45–63 (1939)]. *M. H. Stone* (Cambridge, Mass.).

Alaoglu, Leon. Weak topologies of normed linear spaces. Ann. of Math. **41**, 252–267 (1940). [MF 1014]

By using the notion of Moore-Smith limits over directed sets the author removes the condition of separability in several important theorems of Banach; for example, the unit sphere in an adjoint space is bi-compact in the weak topology of functionals; the notions of transfinite closure, regular closure, and weak closure are equivalent for linear manifolds. If one replaces the notion of unconditional convergence in G. Birkhoff's theory of integration by weak unconditional convergence (either as points or as functionals) no new difficulties arise. The author discusses the general notion of a multiple valued weak derivative and

derives most of the known results on differentiation of vector valued functions from it. *N. Dunford.*

Boas, R. P., Jr. Some uniformly convex spaces. Bull. Amer. Math. Soc. 46, 304–311 (1940). [MF 1838]

The author gives a short proof of certain inequalities due to Clarkson by means of which Clarkson proved the uniform convexity of L^p and l^p, $p>1$. The proof uses M. Riesz's convexity theorem for linear forms and is generalized to show that $L^p\{L^q\}$, $L^p\{l^q\}$, $l^p\{L^q\}$ and $l^p\{l^q\}$ are all uniformly convex for $p>1$, $q>1$. The author also gives an elementary proof of one of Clarkson's inequalities.

N. Levinson (Cambridge, Mass.).

Šmulian, V. On some geometrical properties of the sphere in a space of type (B). C. R. (Doklady) Acad. Sci. URSS (N.S.) 24, 648–652 (1939). [MF 2054]

Šmulian, V. On some geometrical properties of the unit sphere in the space of the type (B). Rec. Math. N.S. [Mat. Sbornik] 6 (48), 77–94 (1939). (Russian. English summary) [MF 1436]

Let E be the normed linear space of bounded functions $x(q)$ defined on a set Q, where $\|x\|=\sup_{q\epsilon Q}|x(q)|$. A sequence $\{q_n\}\subset Q$ is called extremal for $x_0(q)$ if $\|x_0\|=|\lim_{n\to\infty} x_0(q_n)|$. The author proves the following two theorems: (1) In order that $\|x\|$ in E be weakly differentiable at $\|x_0\|$ ($\lim_{h\to 0}(\|x_0+hx\|-\|x_0\|)/h$ exists for every $x\epsilon E$), it is necessary and sufficient that for every sequence $\{q_n\}$ extremal for $x_0(q)$ and for every $x\epsilon E$ there exist $\lim_{n\to\infty}(x(q_n)\cdot x_0(q_n))$, the limit being independent of the particular choice of extremal sequence. (2) In order that the space be strictly normalized ($\|x+y\|=\|x\|+\|y\|$ implies $x=t\cdot y$ if $x, y\neq 0$), it is necessary and sufficient that for every pair $x, y\epsilon E$ with a common extremal sequence one have $x=t\cdot y$. In case E is such that every $|x(q)|$ attains its maximum, then in (2) one may substitute "attaining their maxima at a common point" for "with a common extremal sequence." These theorems are applied to the spaces (C), (c_0), (m) and then, for arbitrary Banach space E, to E and $\bar{E}$ by taking Q as the unit sphere of $\bar{E}$ and E, respectively. The relationship between weak differentiability and strict normalization for any Banach space is considered and the following two theorems and various refinements of them are proved: (3) If the unit sphere in E is weakly compact, then for $\|f\|$, $f\epsilon\bar{E}$, to be weakly differentiable it is necessary and sufficient that E be strictly normalized. (4) If every linear functional $f(x)$ attains its maximum on the unit sphere of E, then for strict normalization of $\bar{E}$ it is necessary and sufficient that $\|x\|$ be weakly differentiable. For a limiting ordinal ϑ notions of ϑ-completeness (similar to transfinite completeness) and of ϑ-separability are introduced and exploited; for example, if the space is ϑ_0-separable and ϑ-complete for every $\vartheta\leqq\vartheta_0$, then it is reflexive. *J. V. Wehausen* (New York, N. Y.).

Hadwiger, H. Über das Umordnungsproblem im Hilbertschen Raum. Math. Z. 46, 70–79 (1940). [MF 1480]

A sequence $A=(a_1, a_2, \cdots$, ad inf.) of elements of Hilbert space is considered, for which $\lim_{n\to\infty} a_n=0$; A_n denotes the set of the first n elements of A, that is, $A_n=\{a_1, a_2, \cdots, a_n\}$. Furthermore, let $[A_n]$ denote the sum $a_1+a_2+\cdots+a_n$. Let S be the set of points of Hilbert space defined as follows: a point a is contained in S if and only if a rearrangement B of A can be found such that a is a limit point of the sequence $\{[B_n]\}$ ($n=1, 2, \cdots$, ad inf.). Finally U is the set of all points a for which a rearrangement B of A can be found such that $\lim_{n\to\infty}[B_n]=a$. It is obvious that U is a subset of S. The author proves: (I) S is closed. (II) If u, v, w are elements of S, then also $u+v-w$ is an element of S. All other results are of negative character. While in a vector space of a finite number of dimensions $U\equiv S$ and S is a linear manifold, in Hilbert space neither of these statements is necessarily true. The author gives an example in which S is not a linear manifold. *A. Wald.*

Lorch, Edgar R. Means of iterated transformations in reflexive vector spaces. Bull. Amer. Math. Soc. 45, 945–947 (1939). [MF 788]

The author shows that von Neumann's mean ergodic theorem holds in a reflexive space. A more general result has been established by Yosida [Proc. Imp. Acad. Tokyo 14 (1938)] and Kakutani [ibid.]. The proof is slightly different from that used by F. Riesz [J. London Math. Soc. 13 (1938)] for the spaces L^p and by Yosida and Kakutani.

B. J. Pettis (Cambridge, Mass.).

Niemytzki, V. Sur les systèmes de courbes remplissant un espace métrique. (Généralisation des théorèmes de Birkhoff.) Rec. Math. N.S. [Mat. Sbornik] 6 (48), 283–292 (1939). (French. Russian summary) [MF 1358]

A definition of the "geometric continuity" of a family of curves is given; this may be shown to be equivalent to "regularity" [H. Whitney, Ann. of Math. 34, 244 (1933)]. The "dynamical limit" E_f of a curve f is defined (following Birkhoff) as the set of all limits of sequences $\{p_i\}$, where $p_i=f(p, t_i)$, $t_i\to\pm\infty$. Theorem 1, that E_f is invariant, is almost trivial. Theorem 2 is that if the space is complete, and f is not closed, then $E_f=\overline{f}-f$. (The proof is unnecessarily complicated, and contains an unessential error.) The paper ends with some theorems on stability in the "senses of Lagrange and Poisson." *H. Whitney.*

Kershner, Richard. Ergodic curves and the ergodic function. Amer. J. Math. 62, 325–345 (1940). [MF 1768]

Let M be a bounded subset of the Euclidean plane. It is known [M. H. Martin, Amer. J. Math. 58, 727 (1936); Bull. Amer. Math. Soc. 43, 541 (1937)] that, corresponding to $\epsilon>0$, there exists a continuous rectifiable curve $C(\epsilon)$ of length $\Lambda(\epsilon)$ such that for every point p of M there is a point q of $C(\epsilon)$ with the distance $pq\leqq\epsilon$, $C(\epsilon)$ has minimum length with respect to all rectifiable curves with this approximating property, and $\Lambda(\epsilon)$ is a continuous function of ϵ. In this paper there are derived a number of properties of $C(\epsilon)$ of which the following are some: (1) $C(\epsilon)$ has no double points; (2) at any non-end point of $C(\epsilon)$ there is a right and left tangent to $C(\epsilon)$; (3) $C(\epsilon)$ has only a countable set of corners; (4) $C(\epsilon)$ has no cusps. In addition it is shown by a simple proof that $\lim_{\epsilon\to 0} 2\epsilon\Lambda(\epsilon)=\text{meas.}\ \bar{M}$.

G. A. Hedlund (Charlottesville, Va.).

Hedlund, Gustav A. A new proof for a metrically transitive system. Amer. J. Math. 62, 233–242 (1940). [MF 1760]

A new proof is given for the metrical transitivity of the geodesic flow on suitably restricted surfaces of constant negative curvature. The proof is essentially elementary and involves little besides the notion of metric density and representation of the surface in terms of suitable coordinates. No use is made of the symbolic methods previously used by the author [Ann. of Math. 35, 787–808 (1934)],

nor of the harmonic functions utilized by E. Hopf [Trans. Amer. Math. Soc. **39**, 299–314 (1936)] in obtaining this result. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Hopf, Eberhard. Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung. Ber. Verh. Sächs. Akad. Wiss. Leipzig **91**, 261–304 (1939). [MF 1147]

Among other results, this paper gives remarkably simple proofs of the ergodicity (metric transitivity) and mixture property of the geodesic flows on a class of n-dimensional manifolds ($n\geqq 2$) of constant negative curvature and the first proof of the ergodicity of the geodesic flow on a class of surfaces of variable negative curvature. The n-dimensional manifolds are defined by a procedure analogous to the familiar one used in the case $n=2$ by assigning the metric

$$ds^2=\frac{4dx_idx_i}{(1-x_ix_i)^2} \tag{A}$$

to the interior of the unit sphere $x_ix_i=1$ and identifying points congruent under a properly discontinuous group F of rigid motions of the hyperbolic geometry defined by (A). The directed geodesics on the manifold M thus obtained define a measure preserving flow T_s in the space Ω of directed elements on M. The author divides these manifolds M into two classes. With M suitably metrized, a geodesic ray goes to infinity if only points of a finite segment of the ray lie in any given bounded set on M. M is of the first class if the set of rays with the same initial point in M and which go to infinity have a zero set of initial directions. This property is independent of the choice of initial point. If M is not of the first class, it is of the second class. In connection with the classic division of groups into first and second kind, it is easy to show that (1) if F is of the second kind, M is of the second class; (2) if F is of the first kind and has a finite set of generators, M is of the first class; (3) if F is of the first kind and has an infinite set of generators, M may be of the second class.

The following theorems concerning the manifolds M are proved: (I) if M is of the second class, almost all the geodesic rays which emanate from any point of M go to infinity (the flow is dissipative); (II) if M is of finite volume, the geodesic flow T_s is ergodic (metrically transitive); (III) if M is of the first class but not necessarily of finite volume, T_s is ergodic [for the definition in this case cf. E. Hopf, Ergodentheorie, §14]; (IV) if M is of finite volume, T_s is a mixture. The results (I), (II) and (IV) are known for the case $n=2$, but the methods of proof of (I) and (II) are especially simple compared with any previously attained. The germ of the idea rests in the observation that asymptotic geodesic rays must display the same behavior with respect to going to infinity or distribution in Ω and in the fact that, if a set of oriented geodesics displays one type of behavior when traced out in the positive sense, then, except for a set of measure zero, the members of the set will display the same behavior when traced out in the opposite sense. The proof of (IV) is obtained by simplification of Hedlund's proof involving horocycles [cf. Ann. of Math. **40**, 370] for the case $n=2$ and then generalization to the case $n>2$.

Finally, the author proves the ergodicity of the geodesic flow for a class of surfaces (of class C^∞) of variable negative curvature. It is assumed that the Gaussian curvature K of the surface S lies between two negative constants and that the directional derivative of K is uniformly bounded. A division of the surfaces S into classes is effected by definitions similar to those made with respect to the manifolds M. It is shown that if S is of the first class the geodesic flow is ergodic, while if S is of the second class the flow is dissipative. This implies in particular that the geodesic flow on any closed analytic surface of negative curvature is metrically transitive. The proofs necessitate an analysis of the properties of asymptotic geodesic rays. *G. A. Hedlund.*

Calculus of Variations

Tricomi, Francesco. Determinazione delle estremali di un certo integrale. Atti Accad. Sci. Torino **75**, 97–100 (1939). [MF 1885]

Integration of the Euler equation corresponding to

$$\int_0^\omega yy'\left(\frac{y'^2}{1+y'^2}-\sin^2\alpha\right)^3dx,$$

where α and ω are constants. The equation does not contain x explicitly. The integral is said to occur in ballistics.

W. Feller (Providence, R. I.).

Mancill, Julian D. On the Carathéodory condition for unilateral variations. Bull. Amer. Math. Soc. **46**, 363–366 (1940). [MF 1849]

The author modifies a proof, previously given by the reviewer, of the condition that the determinant

$$\left|\frac{\partial\varphi^\alpha}{\partial a^i}\,\frac{\partial\varphi^\alpha}{\partial t}\right|$$

of an $(n-1)$-parameter family of extremaloids $x^\alpha=\phi^\alpha(t,a)$ $(\alpha=1,\cdots,n)$ cannot change sign at the corners of the minimizing extremaloid, in such a way that it becomes applicable to unilateral variations in the plane.

L. M. Graves (Chicago, Ill.).

Cole, Nancy. The index theorem for a calculus of variations problem in which the integrand is discontinuous. Amer. J. Math. **62**, 249–276 (1940). [MF 1762]

The purpose of this paper is to establish the Morse index theorem for an ordinary variational problem in parametric form in m-space in which the integrand is ordinarily of class C''' but may be discontinuous with one-sided limits when the point (x) rests on certain deflecting $(m-1)$ manifolds M_i of class C^2. The basic curve g is a broken extremal with a finite number of corners P_i resting on the respective manifolds M_i, with M_i not tangent to either arc of g at P_i. The curve g is assumed to satisfy the "primary incidence" relations. These relations are necessary conditions at the corners P_i that g afford a minimum to J relative to neighboring admissible curves of class D'. A broken extremal with corners at most on the manifolds M_i, and there satisfying the primary incidence relations, is termed an extremaloid. Conjugate points and their orders are then defined in terms of extremaloids and the index theorem is stated in the expected way. The proofs involve a study of the second variation. A generalization of the Sturm separation theorem is obtained. References are to Bolza, Mason and Bliss, Graves, Miles and Morse. *M. Morse* (Princeton, N. J.).

McShane, E. J. An estimate of the Weierstrass $\mathcal{E}$-function. Ann. of Math. **41**, 314–320 (1940). [MF 1814]

This paper is concerned with an inequality satisfied by

the $\mathcal{E}$-function for a general problem of the calculus of variations of Bolza type whenever the strengthened Weierstrass condition holds. No condition of non-singularity is involved in the proof presented, and hence for Bolza problems the inequality is established under conditions comparable to those which suffice for free problems. The proof previously given by the reviewer [Ann. of Math. **38**, 662–678 (1937)] of this inequality for problems of Bolza type made important use of the additional assumption of non-singularity. *W. T. Reid* (Chicago, Ill.).

McShane, Edward James. Curve-space topologies associated with variational problems. Ann. Scuola Norm. Super. Pisa (2) **9**, 45–60 (1940). [MF 1752]

The author considers six definitions of the neighborhood of a curve which may be used to define weak minima in parametric problems of the calculus of variations. The paper catalogues the relations between neighborhoods defined by means of the various definitions. The nature of the results he obtains is evident from the following two examples. If a curve is of class D^1 any neighborhood in length (a neighborhood under the sixth definition given) contains a neighborhood in direction (a neighborhood under the second definition given). However, no sufficiently small neighborhood in direction contains any neighborhood in length, if the curve whose neighborhood is being considered is rectifiable and nondegenerate. Under any of the definitions the space of nondegenerate curves of class D^1 is separable, and under three of these definitions this space is complete. Under the other three definitions every functional defined by an integral of the calculus of variations is continuous on this space of curves. *C. B. Tompkins* (Princeton, N. J.).

Douglas, Jesse. Theorems in the inverse problem of the calculus of variations. Proc. Nat. Acad. Sci. U. S. A. **26**, 215–221 (1940). [MF 1607]

This note is a summary of results to be published in detail later. The families $\mathfrak{F}$ of curves in three-space which constitute the solutions of equations

$$y''=F(x, y, z, y', z'),\quad z''=G(x, y, z, y', z')$$

are to be classified as extremal or non-extremal; a family is extremal if it consists of the extremals of an integral of the calculus of variations. A set of nine functions A, B, C, A_1, B_1, C_1, A_2, B_2, C_2 of the variables x, y, z, y', z' are defined in terms of F, G and their derivatives; the definitions are too long for us to set forth here. Four cases are distinguished, according as the rank of the matrix of the nine functions is 0, 1, 2 or 3. In the first case the family is extremal; in the integrand there are two arbitrary functions of three variables and two of two variables. In the fourth case the family is non-extremal. In the third case, let Δ_1, Δ_2, Δ_3 be the cofactors of A_2, B_2, C_2 in the determinant of the nine functions. If $\Delta_1\Delta_3-\Delta_2^2\neq 0$, the family is extremal if and only if a certain calculable differential expression is an exact differential. The integrand is then essentially uniquely determined. Case II is also studied, and several sets of conditions are shown to be sufficient in order that the family be extremal; in each case the degree of arbitrariness of the integrand is stated. *E. J. McShane* (Charlottesville, Va.).

Douglas, Jesse. The analytic prolongation of a minimal surface across a straight line. Proc. Nat. Acad. Sci. U. S. A. **25**, 375–377 (1939). [MF 885]

Let M be a minimal surface, the coordinates x, y and z being the real parts of functions analytic in the upper half of the w-plane and having the sum of the squares of their derivatives equal to zero. Let an interval $a<u<b$ of the real axis correspond to a line l in three-space, in the sense that as w approaches any fixed point of (a, b) the distance of the corresponding (x, y, z) from the line l tends to zero. By rotating M through 180° about l we obtain a minimal surface M'. The surface $M+M'$ is then a single minimal surface, regular where it traverses l. This is a slight generalization of a result previously established by the author, with simplification of proof. *E. J. McShane.*

Courant, R. and Davids, N. Minimal surfaces spanning closed manifolds. Proc. Nat. Acad. Sci. U. S. A. **26**, 194–199 (1940). [MF 1603]

Given in xyz-space a closed surface M of genus $p>0$ and a simple closed polygon H which has no points in common with M and is linked with M (that is, H is linked with every member of a class of equivalent non-bounding cycles on M). The problem solved in this paper consists of showing the existence of a simply connected minimal surface of least area whose boundary lies on M and is linked with H. While in a general way the method is similar to that used in a previous note of Courant [The existence of a minimal surface of least area bounded by prescribed Jordan curves and prescribed surfaces, Proc. Nat. Acad. Sci. U. S. A. **24** (1938)], it appears that the boundary condition considered in the present paper makes it necessary to introduce various new devices in the course of the existence proof.

T. Radó (Columbus, Ohio).

Menger, Karl. A theory of length and its applications to the calculus of variations. Proc. Nat. Acad. Sci. U. S. A. **25**, 474–478 (1939). [MF 879]

Let a real number $d(p, q)$ be associated with any two points of a certain limit class S. Designate by I, II, $\cdots$ the following conditions: I. $d(p, p)=0$ for every p. II. For every $\epsilon>0$ there exists an $\eta>0$ such that $|d(p, q)-d(p', q')|<\epsilon$ if $|d(p, p')|+|d(q, q')|<\eta$. III. (referring to two different functions d and d_1) For every $\epsilon>0$ there exists an $\eta>0$ such that $|d_1(p, q)|<\epsilon$ for $|d(p, q)|<\eta$. Let $P=(p_1, \cdots, p_n)$ be any ordered set of points and put $\lambda(P; d)=\sum_{i=1}^{n-1}d(p_i, p_{i+1})$ and $|\lambda|(P; d)=\sum|d(p_i, p_{i+1})|$. IV. For every $\epsilon>0$ there exists an $\eta>0$ such that $\lambda(P; d)\geqq d(p_1, p_n)-\epsilon|d(p_1, p_n)|$ for $\max d(p_i, p_{i+1})<\epsilon$.

With these notations let there now be given two distance functions δ and δ_1, δ satisfying I, II and δ_1 satisfying IV and such that with $\delta=d$ and $\delta_1=d_1$ condition III is satisfied. C will be called a continuous curve if there is a mapping $\gamma\rightarrow p(\gamma)$ of the closed interval $[0, 1]$ onto C such that $\delta(p(\gamma_\nu), p(\gamma))\rightarrow 0$ for $\gamma_\nu\rightarrow\gamma$. Let P^ν be a polygon inscribed in C, that is, a finite set of points $p_i^\nu=p(\gamma_i^\nu)$, $i=1, \cdots, n_\nu$, with $\gamma_i^\nu<\gamma_{i+1}^\nu$. A sequence of such polygons P^ν is called distinguished if $\lim_\nu \max_i (\gamma_{i+1}^\nu-\gamma_i^\nu)=0$, and $\{P^\nu\}$ is called maximal if $\lambda_1(C)=\lim_\nu \lambda(P_\nu, \delta_1)=\text{l.u.b.}_{\{\bar{P}_\nu\}} \limsup_{\nu\to\infty} \lambda(\bar{P}_1^\nu; \delta_1)$, where $\{\bar{P}^\nu\}$ traverses all distinguished sequences. The first main result is: If on C a maximal sequence $\{P^\nu\}$ exists for which $\text{l.u.b.}\ |\lambda|(P^\nu; \delta_1)/\lambda(P^\nu; \delta_1)<\infty$ for $\lambda_1(C)<\infty$ and $\text{l.u.b.}\ |\lambda|(P^\nu, \delta_1)|<\infty$ for $\lambda_1(C)=0$, then $\lambda(C; \delta_1)=\lim \lambda(Q^\nu; \delta_1)$ will exist (possibly equaling $\pm\infty$) and be the same for all distinguished sequences $\{Q^\nu\}$. Furthermore, $\lambda(C; \delta_1)$ is lower semi-continuous if the Fréchet distance derived from δ is used for the curves. Let δ_2 be another distance function satisfying III (with $d=\delta$ and $d_1=\delta_2$) and IV and satisfying IV_{12}: For any two numbers $\epsilon_1>0$ and $\epsilon_2>0$ there exists an $\eta>0$ such that for every polygon $P=(p_1, \cdots, p_n)$ with $\max_{i,j} \delta(p_i, p_j)<\eta$ one has $\lambda(P; \delta_1)$

$\geqq \delta_1(p_1, p_n) - \epsilon|\delta_1(p_1, p_n)| - \epsilon[|\delta_2(p_1, p_n)| + |\lambda|(P, \delta_2)]$. Then the function $\lambda(C; \delta_1)$ will be lower semi-continuous on the set of all curves for which $\lambda(C, \delta_2) \leqq M$ for every $M < \infty$. Assume furthermore that for any two distinct points p and q and every polygon of the form $P = (p = p_1, p_2, \cdots, p_n = q)$ one has $\lambda(P; \delta_2) > C(p, q) > 0$, and that for every $\epsilon > 0$ an $\eta > 0$ exists such that a curve from p to q with $\lambda(C; \delta_2) < \epsilon$ exists as soon as $\delta(p, q) < \eta(p, \epsilon)$. Then in a compact space (with respect to δ) the set of all curves C with $\lambda(C; \delta_2) \leqq r$ is compact. If finally $\lambda(C; \delta_2)$ is bounded for all curves for which $\lambda(C; \delta_1)$ remains below a bound N, then every closed class K of curves C with finite $\lambda(C; \delta_2)$ contains a limit element of a minimizing sequence for $\lambda(C; \delta_1)$ on K; and this limit element minimizes $\lambda(C; \delta_1)$ on K.

H. Busemann (Swarthmore, Pa.).

Theory of Probability

Koopman, B. O. The axioms and algebra of intuitive probability. Ann. of Math. **41**, 269–292 (1940). [MF 1811]

The author axiomatizes intuitive probability notions, reserving the study of the relation of these notions to frequency to a later paper. The notions axiomatized are the intuitive ones (of an individual) "prior to objective experience," and there is no discussion of their relation to actual events. In the axiomatization, eventualities become elements of a Boolean ring with units 0, 1. The concept a if h (a/h) becomes a certain remainder class in this ring. These remainder classes are partially ordered. A numerical theory of probability is then set up which reproduces in a rough way the original probability relations. (Thus eventualities which have probability 1 may not be equiprobable.) This numerical theory is developed using the assumption that, for every positive integer n, there is a set of n mutually incompatible eventualities $u_1, \cdots, u_n$, whose sum is not null, such that the (u_j/u) are equiprobable. An outer measure $p^*(a/h)$ is defined as the limit $(n \to \infty)$, which is shown to exist, of the minimum value of T/n for which (a/h) is not more probable than $(u_1$ or $\cdots$ or $u_T)/u$. An inner measure $p_*(a/h)$ is also defined, and, if $p_* = p^*$, (a/h) is called appraisable, with probability $p_* = p^*$. The probability numbers have the usual additivity properties, and, by representation theorems of Garrett Birkhoff and Stone, the probability numbers can be considered as defining an additive set function on a system of sets of some space.

J. L. Doob (Urbana, Ill.).

Onicescu, Octav. La probabilité d'un événement isolé. Acad. Roum. Bull. Sect. Sci. **22**, 280–286 (1940). [MF 1699]

Onicescu, Octav. La définition de la probabilité et le problème de la roulette. Acad. Roum. Bull. Sect. Sci. **21**, 106–109 (1939). [MF 1506]

Fréchet, Maurice. Conditions d'existence de systèmes d'événements associés à certaines probabilités. J. Math. Pures Appl. **19**, 51–62 (1940). [MF 1991]

The author considers the following: the system consisting of the probabilities of various conjunctions of r events selected from m; the system consisting of sums of such probabilities, each sum being extended over all possible conjunctions associated with a given r; the system consisting of the probabilities of the various atoms, where an atom is a conjunction of r events and the negatives of the remaining $n-r$ events; the system of the probabilities that exactly r events will occur $(r = 1, 2, \cdots, m)$; the system of probabilities that at least r will occur. The author determines the conditions under which a given system of numbers can be taken as the probabilities within one of these systems. It is understood that the number of numbers in a system is equal to the number of probabilities in the corresponding system. He shows that the probabilities of the atoms are subject only to the conditions that they are non-negative and that their sum is unity. In the case of the system concerning the conjunction of r events, certain latitude is possible. In fact for a given r the various conjunctions can be assigned equal probabilities. For the system consisting of sums of probabilities of conjunctions, the given events can under certain circumstances be taken as independent. Furthermore, it is always possible to find a system of fictitious independent events which may have complex probabilities but which formally satisfy the conditions.

A. H. Copeland (Ann Arbor, Mich.).

Fréchet, Maurice. Compléments à un théorème de T. S. Broderick concernant les événements dépendants. Proc. Edinburgh Math. Soc. (2) **6**, 109–113 (1939). [MF 1524]

The author considers functions which transform sets of events into single events. He derives a formula for the probabilities of such event functions, proves the uniqueness and observes that in this formula probabilities can be generalized to arbitrary additive (positive or negative) functions. In addition to results which are interesting in themselves, this development serves as a rigorous basis for the generalization of King's formula derived in the paper of Broderick and for certain symbolic expansions which appear in a paper by Jordan. A rigorous development of the generalization of King's formula and of the symbolic expansion of Jordan were, however, obtained by Copeland [Amer. Math. Monthly **44** (1937)] by an entirely different method.

A. H. Copeland (Ann Arbor, Mich.).

Stevens, W. L. Solution to a geometrical problem in probability. Ann. Eugenics **9**, 315–320 (1939). [MF 1459]

On the circumference of a circle of unit length n arcs, each of length x, are marked off at random; the author calculates the probability that every point of the circle is included in at least one of these arcs. *W. Feller.*

Selberg, Henrik L. Über eine Verschärfung der Tchebycheffschen Ungleichung. Arch. Math. Naturvid. **43**, 30–32 (1940). [MF 1395]

Consider a distribution function $F(x)$ with $F(0) = 0$, mean value m and second moment M. Using Schwarz's inequality, the author shows that for $a > s$ we have $1 - F(a) \leqq (M - m^2)/\{(a-m)^2 + M - m^2\}$. *W. Feller.*

Dieulefait, C. On incomplete moments. Revista Ci., Lima **41**, 543–547 (1939). (Spanish) [MF 1645]

The "incomplete" characteristic function for the case of repeated trials with constant probability p is

$$(1) \qquad \varphi(\lambda, t) = \sum_{v=0}^{\lambda} \binom{n}{v} p^v (1-p)^{n-v} e^{vt}, \qquad \lambda < n.$$

The author makes in (1) the substitution

$$\binom{n}{v} = \frac{1}{2\pi i} \int_C \frac{(z+1)^n}{z^{v+1}} dz$$

and obtains

$$(2)\qquad \varphi(\lambda, t) = -\frac{(1-p)^{n-\lambda}(pe^t)^{\lambda+1}}{2\pi i}\int_C \frac{(z+1)^n dz}{(qz-pe^t)z^{\lambda+1}},$$

where the closed curve C surrounds the origin and leaves in its exterior the point $p/(1-p)$. He then multiplies the right member of (2) by e^{-npt} and obtains the characteristic functions for the deviations $v-np$. *A. González Domínguez.*

Geppert, M. P. Su una classe di distribuzioni in due variabili casuali. Giorn. Ist. Ital. Attuari 10, 225–228 (1939). [MF 1891]

The author continues his inquiry concerning conditions under which the law of Bravais is the unique law of distribution in two random variables which (1) conserves the linearity of the regression independent of the orientation of the Cartesian axes and (2) presents the independence of the two variables for a particular choice of axes. It is here shown that the first condition requires that $f(x, y)$ be of the form $f(ax^2+2bxy+cy^2)$, $ac>0$, from which it follows that the second condition is satisfied. *A. A. Bennett.*

Smirnoff, N. Sur les écarts de la courbe de distribution empirique. Rec. Math. N.S. [Mat. Sbornik] 6 (48), 3–26 (1939). (Russian. French summary) [MF 1432]

Let X be a random variable having the distribution function $F(x)$ and let $x_1 \leqq x_2 \leqq \cdots \leqq x_n$ be n observed values of X; $S_n(x)$ denotes the step-function, $S_n(x)=k/n$ for $x_k \leqq x < x_{k+1}$, $S_n(x)=0$ for $x<x_1$, and $S_n(x)=1$ for $x \geqq x_n$. The author investigates problems of the following type: what is the probability that the number of intersections of the "curves" $y=S_n(x)$ (the "empirical" distribution) and $y=F(x)$ (the "theoretical" distribution) is not greater than $tn^{\frac{1}{2}}$. Denoting this probability by $\Phi_n(t)$, it is shown that $\Phi_n(t) \to 1-\exp(-t^2/2)$. The problem can be stated in non-statistical terms also. If $\delta_k(t)=1$ for $(k-1)/n \leqq t \leqq k/n$ and $\delta_k(t)=0$ otherwise, it is easy to see (and it is implicitly contained in the author's work) that

$$\Phi_n(t) = n!\int_{\Omega_n}\cdots\int dT(x)\cdots dT(x_n),$$

where Ω_n is defined by the inequalities $-\infty < x_1 \leqq x_2 \cdots \leqq x_n$, $\delta_1(F(x_1))+\cdots+\delta_n(F(x_n)) \leqq tn^{\frac{1}{2}}$. A simple change of variables shows that $\Phi_n(t)$ does not depend on $F(x)$ (Lemma 1) and therefore it may be assumed that X is uniformly distributed in (0, 1). The proof, which is rather elaborate, makes use of characteristic functions, and the main computational difficulty is caused by the fact that one has to operate in the "symmetric" product $0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_n \leqq 1$ and not in the n-dimensional cube. Some generalizations of the above asymptotic formula are given which seem to be of interest in practical statistics. *M. Kac* (Ithaca, N. Y.).

Dedebant, Georges, Moyal, José et Wehrlé, Philippe. Sur les équations aux dérivées partielles que vérifient les fonctions de distribution d'un champ aléatoire. C. R. Acad. Sci. Paris 210, 243–245 (1940). [MF 1596]

Let X_t be a chance variable, for each value of t, and suppose that X_t has a first derivative in t, in the sense that $(X_{t+h}-X_t)/h \to \dot{X}_t$ in the mean. Then it is shown that, if X_t has a density of distribution $\rho(x, t)$, and if $E[x; \dot{X}_t]$ is the conditional expectation of $\dot{X}_t$ if X_t has the value x,

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}\{E[x; \dot{X}_t]\rho\}=0.$$

This differential equation corresponds to the equation of continuity in the study of flows. A second order equation is also found, and the case in which X_t is an n-dimensional vector is treated. The differential equations are used to derive specific distributions. *J. L. Doob* (Urbana, Ill.).

Loève, Michel. Nouvelles classes de lois limites. C. R. Acad. Sci. Paris 210, 202–204 (1940). [MF 1633]

A distribution function (d.f.) $F(x)$ is said to be divisible by a d.f. $F'(x)$ if there is a d.f. $F''(x)$ such that $F(x) = \int F''(x-\xi)dF'(\xi)$. Let $X_1, X_2, \cdots$ be mutually independent chance variables, and let $S_n = X_1+\cdots+X_n$. Suppose that $\{a_n\}$ is a sequence of positive constants, and that, for some increasing sequence of values of n, (a) $a_n/a_{n+1} \to \alpha$ $(0<\alpha<1)$ and (b) the d.f. of S_n/a_n converges to a d.f. $F(x)$ at the points of continuity of $F(x)$. Then (A) the d.f. $F(x)$ is divisible by $F(x/\alpha)$. Conversely, if (A) is true for a d.f. $F(x)$, $F(x)$ can be obtained as just described. The property (A) is true for a d.f. $F(x)$ if and only if it is the d.f. of a sum of the form $\xi_0+\alpha\xi_1+\alpha^2\xi_2+\cdots$, where the ξ_j are mutually independent chance variables with the same distributions. *J. L. Doob* (Urbana, Ill.).

Sternberg, Wolfgang. The general limit theorem in the theory of probability. Bull. Amer. Math. Soc. 46, 292–298 (1940). [MF 1836]

A proof of the central limit theorem of the theory of probability based on the ideas of Petrowsky and Kolmogoroff [see A. Khintchine, Asymptotische Gesetze der Wahrscheinlichkeitsrechnung, Ergebnisse der Mathematik und ihrer Grenzgebiete, Berlin, 1933] is given. To make the exposition simpler the author does not choose the most general conditions and works with the third moments. The upper- and lower-functions which occur in the Petrowsky-Kolmogoroff proof are eliminated. *M. Kac.*

Feldheim, E. Nuova dimostrazione e generalizzazione di un teorema di calcolo delle probabilità. Giorn. Ist. Ital. Attuari 10, 229–243 (1939). [MF 1892]

Let X be a random variable with Prob $\{X=k\} = \binom{n}{k}p^k(1-p)^{n-k}$, where $0<p<\frac{1}{2}$, and let

$$\Delta = \text{Prob}\ \{X \leqq np-1\} - \text{Prob}\ \{X > np\}.$$

Putting $np=[np]+s$, it is shown that, as $n\to\infty$,

$$\Delta = \frac{1-2p-6s}{3\{2\pi np(1-p)\}^{\frac{1}{2}}}+O(n^{-\frac{1}{2}}).$$

If np is an integer $(s=0)$, this is said to reduce to a formula of Simmons [Proc. London Math. Soc. 26, 290–334 (1895)]. *W. Feller* (Providence, R. I.).

Rodgers, Eric. Probable error for Poisson distributions. Phys. Rev. 57, 735–737 (1940). [MF 1803]

The Poisson probability distribution $W_n = x^n e^{-x}/n!$, approximating the probability of obtaining n occurrences of an event, where x is the expected number of those occurrences, is known to approach the normal probability distribution with $x\to\infty$. The author shows that, for Poisson distributions with $x\to\infty$, the probable error and the probability of a deviation not exceeding one standard deviation tend to the corresponding values for a normal distribution. *Z. W. Birnbaum* (Seattle, Wash.).

Onicescu, O. et Mihoc, G. Sur les sommes de variables enchainées dans le cas d'un ensemble numérable de

valeurs. Acad. Roum. Bull. Sect. Sci. **22**, 231–236 (1940). [MF 1702]

Let $x_1, x_2, \cdots$ be chance variables of a Markoff chain (that is, x_n is a function of the states which the given system assumes on the nth trial) corresponding to a system which can take on infinitely many states. The authors prove, using characteristic functions, that under certain hypotheses, the distribution of $x_1+\cdots+x_n$ is nearly normal, a fact proved (using a different method) under somewhat weaker hypotheses by Doeblin [Thèse, Paris, or Bull. Math. Soc. Roum. Sci. **39** (1937)]. *J. L. Doob* (Urbana, Ill.).

Onicescu, Octav et Mihoc, Gh. Sur l'application des équations fonctionnelles de Chapman et Smoluchowsky dans la théorie des chaînes de Markoff. Acad. Roum. Bull. Sect. Sci. **21**, 110–112 (1939). [MF 1507]

Let $x_0, x_1, x_2, \cdots$ be the chance variables of a Markoff chain, with $|x_n|\leqq 1$; there is given a probability density $\varphi(x, y)$ that x_{n+1} will be y if $x_n=x$. Let $\Phi(x, y; n; t)$ be the characteristic function of the distribution of the sum $x_1+\cdots+x_n$ under the hypothesis that $x_0=x$, and that $x_n=y$. Then it is shown that Φ satisfies the Smoluchowsky equation

$$\Phi(x, y; m+n; t)=\int_{-1}^{1}\Phi(x, z; n; t)\Phi(z, y; n; t)dz.$$

This result is extended to chains in which the transition probability φ, governing the transition from x_n to x_{n+1}, depends on n. *J. L. Doob* (Urbana, Ill.).

Doeblin, W. Sur certains mouvements aléatoires discontinus. Skand. Aktuarietidskr. **1939**, 211–222 (1939). [MF 2005]

Let W be an abstract space and $\mathfrak{K}$ a completely additive field of sets $E\subset W$. Consider a random point moving in W in such a way that, corresponding to any $x\epsilon W$, any $E\epsilon\mathfrak{K}$ and any two moments s, t with $s<t$, there exists a uniquely determined probability $P(x, E, s, t)$ that, starting at time s at the point x, the moving point will at time t be contained in E. In particular, this transition probability is independent of the movement for $t<s$ (stochastic process of Markoff type). Apart from trivial conditions, $P(x, E, s, t)$ is only subjected to the Chapman identity

$$P(x, E, s, t)=\int_W P(y, E, u, t)P(x, dW_y, s, u),$$

valid for $s<u<t$, and to the initial condition that $P(x, E, s, s)$ equals 1 or 0 according to whether $x\epsilon E$ or not. The present paper is concerned with the special class of such processes which is characterized by the condition that $P(x, x, s, t)\rightarrow 1$ uniformly with respect to x and s, as $t\rightarrow s$. This implies in particular that the moving point will, in every finite time interval, undergo only a finite number of changes ("purely discontinuous process"). The author shows that under these conditions there exist two functions $\Pi(x, s, t)$ and $\Psi(x, E, s, t)$, which may, roughly, be described, respectively, as the probability that the moving point will remain in its initial position x during (s, t), and the compound probability that it will undergo a change, the first jump taking it into the set E. These functions satisfy, besides trivial conditions, the relations (i) $\Pi(x, s, t)=\Pi(x, s, u)\Pi(x, u, t)$ and (ii) $\Psi(x, E, s, t)=\Psi(x, E, s, u)+\Pi(x, s, u)\Psi(s, E, u, t)$, both valid for $s<u<t$. The author shows that, conversely, each couple $\Pi(x, s, t)$ and $\Psi(x, E, s, t)$ of functions satisfying (i) and (ii) belongs to a uniquely determined transition probability of the described sort. The argument is based on a direct investigation of the positions in a finite number of moments, and a passage to the limit. *W. Feller* (Providence, R. I.).

Potoček, J. La diffusion et la notion de réversibilité de M. Kolmogoroff. Bull. Int. Acad. Sci. Bohême **1939**, 1–10 (1939). [MF 2137]

In this paper the diffusion in a 3-dimensional bounded region S is treated as a stochastic process, the consideration being restricted to the case where the corresponding "Fokker-Planck equation" [Kolmogoroff, Math. Ann. **104**, 415–458 (1931) and **108**, 149–160 (1933)] takes on the form

$$(*)\qquad \frac{\partial u}{\partial t}=D\sum\frac{\partial^2 u}{\partial x_i^2}-\sum\frac{\partial}{\partial x_i}(f_i(x)u),$$

where D is a constant (homogeneous diffusion under the influence of a stationary force). Using the methods employed in the one-dimensional nonhomogeneous case by Feller [Math. Ann. **113**, 113–160 (1936)], it is shown that, under slight regularity conditions, the Green function $G(t, y, x)$ of the problem can be interpreted as transition probability, and that $G(t, y, x)>0$ in S. This implies, according to Kolmogoroff [cf. the papers quoted above], the existence of a stationary probability distribution $U(x)$ and the validity of the ergodic principle in the sense that, for all $y\epsilon S$, $G(t, y, x)\rightarrow U(x)$ as $t\rightarrow\infty$. It is shown that the process is reversible in the sense of Kolmogoroff [Math. Ann. **113**, 766–772 (1937); cf. also ibid. **112**, 155–160 (1935)] if and only if $Uf_i=D\partial u/\partial x_i$ for $i=1, 2, 3$. Another necessary and sufficient condition is the existence of a function $P(x)$ such that $f_i=\partial P/\partial x_i$, a condition found by Kolmogoroff [cf. the 1937 paper quoted above] in the case where S is a closed Riemannian manifold. *W. Feller* (Providence, R. I.).

Theoretical Statistics

✶Aitken, A. C. Statistical Mathematics. Oliver and Boyd, Edinburgh, 1939. vii+153 pp. 4s/6d.

This book gives a survey of the mathematical basis of statistics from as elementary a standpoint as possible. The subject-matter includes a discussion of probability, the commoner probability and frequency distributions, the elements of correlation, partial correlation and regression, the sampling distribution of the simpler coefficients, and some comments on the problem of estimation. The methods used presume a knowledge of algebra up to the binomial theorem and the simpler infinite series, and of elementary calculus.

Evans, W. D. Note on the moments of a binomially distributed variate. Ann. Math. Statistics **11**, 106–107 (1940). [MF 1618]

It is noted that the second of the triangular tables given by J. A. Joseph [Ann. Math. Statistics **10**, 293 (1939)] gives the coefficients in the expressions for the moments about zero of a binomially distributed variate, and an explicit formula is given for any element in the table and hence for any such coefficient in terms of repeated summations of integers which, in application, is equivalent to building up the table to the desired point. *C. C. Craig.*

Gumbel, E. J. La dissection d'une répartition. Ann. Univ. Lyon (3) A **11**, 39–51 (1939). [MF 1787]

The author solves the dissection problem for frequency

distributions, which are the sums of two asymmetrical exponential distributions or of two Poisson distributions, restricting himself, however, to the case in both instances in which the proportion contributed to the sum by each component is known. *C. C. Craig* (Ann Arbor, Mich.).

Geary, R. C. The mathematical expectation of the mean square contingency when the attributes are mutually independent. J. Roy. Statist. Soc. **103**, 90–91 (1940). [MF 1942]

Suppose the probability associated with the cell C_{ij} in the ith row and jth column of a contingency table is $p_{\cdot j}p_{\cdot i}$, where $\sum_{i=1}^{k} p_{i\cdot}=\sum_{j=1}^{l} p_{\cdot j}=1$. In a sample of size N, let n_{ij} be the number of observations in the cell C_{ij}, and let $n_{i\cdot}=\sum_{j=1}^{l} n_{ij}$, $n_{\cdot j}=\sum_{i=1}^{k} n_{ij}$. The author has shown by direct evaluation that the mean value of the mean square contingency

$$\varphi'^2=\sum_{i,j}\left(n_{ij}-\frac{n_{i\cdot}n_{\cdot j}}{N}\right)^2 \Big/ n_{i\cdot}n_{\cdot j}$$

is $(k-1)(L-1)/(N-1)$. The same result was obtained by Bartlett [Proc. Roy. Soc. London **160**, 280 (1937)] by the use of a generating function. *S. S. Wilks.*

Bartlett, M. S. The standard errors of discriminant function coefficients. Suppl. J. Roy. Statist. Soc. **6**, 169–173 (1939). [MF 1878]

As it was pointed out by Fisher, a linear discriminant function analysis may also be regarded as an analysis by means of a linear regression. The present paper discusses interpretations that may be given to the standard error of a regression coefficient from this point of view. In many applications the value of the dependent variable is known exactly but in some examples it is subject to error also, and it is pointed out that proper allowance for such errors is difficult to provide. In a particular example from genetics, as an example of this fact, an approximate expression for the standard error of the "genetic advance," which is a function of the regression coefficients, is obtained.
C. C. Craig (Ann Arbor, Mich.).

Hoel, Paul G. The errors involved in evaluating correlation determinants. Ann. Math. Statistics **11**, 58–65 (1940). [MF 1610]

Correlation determinants are usually evaluated by the method of Chio using diagonal elements as pivotal in the successive reductions. If in such a series of reductions numerical quantities are rounded off to a fixed decimal accuracy, the author obtains an upper bound for the difference between the end result of this evaluation process and the exact value of the determinant, and this upper bound is expressed in terms of quantities calculable from the determinant itself. Since the errors introduced in the "rounding-off" obey a rectangular distribution and since the error of evaluation depends in a simple way on a linear form in these "rounding-off errors," which for determinants of even moderate dimensions is shown to be approximately normally distributed, it is possible, as it is shown in this paper, to estimate bounds for the error of evaluation with assigned probabilities that on a particular occasion they will be exceeded. *C. C. Craig* (Ann Arbor, Mich.).

Fisher, R. A. The sampling distribution of some statistics obtained from non-linear equations. Ann. Eugenics **9**, 238–249 (1939). [MF 1465]

The first two sections of this paper include brief résumés of the previously published sampling theory of Fisher's z test and his test of significance of a harmonic component as examples of statistics obtained from the solution of non-linear equations. The remainder of the paper is devoted to a brief derivation of the simultaneous sampling distribution of the roots of the equation

$$(1) \qquad |a_{ij}-\theta(a_{ij}+a_{ij}')|=0,$$

where $\|a_{ij}\|$ and $\|a_{ij}'\|$ $(i,j=1,2,\cdots,p)$ are matrices of second order product sums based on $n_1\geqq p$ and $n_2\geqq p$ degrees of freedom, respectively, such that the elements of each matrix are distributed according to Wishart's joint distribution law, the two sets of elements being independent in the probability sense. Several special cases of the general sampling distribution of the roots of (1) are considered, including a derivation of the distribution of the roots of the equation

$$|a_{ij}-\lambda\alpha_{ij}|=0$$

for the case in which $\|\alpha_{ij}\|$ is the matrix of variances and covariances in the normal population underlying the Wishart distribution.

The new results reported in this paper are formally demonstrated in the paper reviewed below. *S. S. Wilks.*

Hsu, P. L. On the distribution of roots of certain determinantal equations. Ann. Eugenics **9**, 250–258 (1939). [MF 1466]

The present paper gives a complete demonstration of Fisher's result mentioned in the above review, and also handles the case in which $p>n_1$. Further, if the $p(p+1)/2$ covariances (about a fixed point) s_{ij} of the p variables are such that $\|s_{ij}\|$ is always non-singular and if their distribution depends only on the latent roots of $\|s_{ij}\|$, the general distribution of the latent roots themselves is deduced. Finally, it is shown that Hotelling's canonical correlations between two sets of random variables are distributed as the square roots of the θ's, the form of the distribution function of the θ's being derived under the condition that only the first set of variables be normally distributed.
C. C. Craig (Ann Arbor, Mich.).

Madow, William G. The distribution of quadratic forms in non-central normal random variables. Ann. Math. Statistics **11**, 100–103 (1940). [MF 1616]

This note contains an extension of a theorem of R. A. Fisher and W. G. Cochran which states necessary and sufficient conditions that a set of quadratic forms in normally and independently distributed random variables should themselves be independently distributed in χ^2-distributions. In the Fisher-Cochran theorem it is assumed that the mean values of the random variables are zero. Here this restriction is eliminated and necessary and sufficient conditions are given that a set of quadratic forms in normally and independently distributed random variables should themselves be independently distributed in χ'^2-distributions. A distribution is called a χ'^2-distribution if it is equal to the distribution of a sum of squares of normally and independently distributed random variables with unit variances without any restriction regarding their mean value.
A. Wald (New York, N. Y.).

Wald, Abraham. A note on the analysis of variance with unequal class frequencies. Ann. Math. Statistics **11**, 96–100 (1940). [MF 1615]

Consider p groups of variables x_{ij}, $i=1,2,\cdots,p$, $j=1,2,\cdots,m_i$, such that $x_{ij}=\eta_i+\epsilon_{ij}$, where η and ϵ are all

independent and normally distributed about unknown means with unknown S.D.'s σ_1 and σ_2, respectively. The problem considered is that of finding the confidence limits for the ratio $\lambda=\sigma_2/\sigma_1$. Let $x_{i.}$ denote the mean of the x's in the ith group and $x_{..}=\sum w_i x_{i.}/\sum w_i$, where $w_i=m_i/(1+m_i\lambda)$. The results of the author are based on the fact that the two sums $\sum w_i(x_{i.}-x_{..})^2/\sigma_1^2$ and $\sum\sum(x_{ij}-x_i)^2/\sigma_1^2$ are independently distributed as χ^2 with $p-1$ and $\sum m_i-p$ degrees of freedom, respectively, so that the distribution of their ratio F is also known. If F_1 and F_2 are the lower and the upper fixed per cent points of the distribution of F, then the confidence limits sought are obtained by solving for λ the equations $F=F_k$ $(k=1, 2)$. When the m_i are unequal, then the solution of those equations is troublesome and the author indicates a method of simplification. *J. Neyman.*

Dressel, Paul L. Statistical seminvariants and their estimates with particular emphasis on their relation to algebraic invariants. Ann. Math. Statistics **11**, 33–57 (1940). [MF 1609]

The first section of this paper is devoted to exploration of the connections between the seminvariants of algebraic theory and the moment functions called seminvariants in mathematical statistics. The latter quantities are shown to be obtained from homogeneous isobaric fractional seminvariants by an application of the Roberts duality theorem. The consideration of complete systems of such seminvariants gives not only the seminvariants of Thiele as one such system but also the central moments as another, and still another system of seminvariants of minimum degree which is of special interest in statistical theory. Linear independence of statistical seminvariants and moment functions invariant under translation and change of scale are discussed in this connection. The second section is principally concerned with the estimation of statistical seminvariants; that is, the calculation of moment functions from samples whose expected values are seminvariants of the population sampled. A systematic means of calculation of such quantities is developed with accompanying tables of results. It is shown that such estimates are in general composite seminvariants and are hence seminvariants themselves. It is shown that an interesting property of the third complete system of seminvariants mentioned above is that they are invariant under estimate, that is, under the process of taking their expected values. *C. C. Craig* (Ann Arbor, Mich.).

Cochran, W. G. Note on an approximate formula for the significance levels of z. Ann. Math. Statistics **11**, 93–95 (1940). [MF 1614]

In this note the author suggests an improvement on R. A. Fisher's approximation formula for obtaining the significance levels of z for the larger values of n_1 and n_2, the numbers of degrees of freedom of the two sums of squares used in z. *S. S. Wilks* (Princeton, N. J.).

van der Waerden, B. L. Vertrauensgrenzen für unbekannte Wahrscheinlichkeiten. Ber. Verh. Sächs. Akad. Wiss. Leipzig **91**, 213–228 (1939). [MF 1188]

For a given confidence coefficient $1-\epsilon$, it is well known [C. J. Clopper and E. S. Pearson, Biometrika **26**, 404–413 (1934)] that conservative confidence limits of the probability p associated with a binomial distribution can be determined for each r by finding the range of values of p for which

$$W(p)=\sum_{r_1\leq r\leq r_2}\binom{n}{r}p^r(1-p)^{n-r}\geqq 1-\epsilon,$$

where r_1 and r_2 are, respectively, the largest and smallest integers such that the sum of binomial terms for $r<r_1$ and for $r>r_2$ are each not greater than $\epsilon/2$. For small values of n, the value of $W(p)$ oscillates considerably as p varies, thus making confidence limits for some values of p unnecessarily conservative. The author suggests the use of "average" confidence limits by choosing r_1 and r_2 for the various values of p so that $\int_0^1 W(p)dp$ is approximately equal to $1-\epsilon$. By using a certain refinement of the Gaussian approximation to the binomial distribution, it is found that, for small values of n, confidence limits can be determined which closely approximate the "average" confidence limits. For large values of n, the author points out that satisfactory confidence limits can be obtained, for a given r, by taking all values of p for which

$$\frac{(r-np)^2}{np(1-p)}<\chi^2,$$

where the value of χ^2 for a given ϵ is determined from the χ^2-distribution law with one degree of freedom. "Average" confidence limits are considered in a similar manner for the Poisson distribution law. For testing the difference between two relative frequencies, the author mentions the usual χ^2 criterion. Finally, a method is outlined for approximating a cumulative distribution function from experimental data by means of confidence limits. Tables to simplify the application of the method are given for $n=1$ to 25 and for $\epsilon=.157$. *S. S. Wilks* (Princeton, N. J.).

Olshevsky, Louis. Two properties of sufficient statistics. Ann. Math. Statistics **11**, 104–106 (1940). [MF 1617]

Koshal, R. S. Maximal likelihood and minimal χ^2 **in relation to frequency curves.** Ann. Eugenics **9**, 209–231 (1939). [MF 1463]

This paper returns to a controversial discussion of the relative merits of three methods as applied to a problem of computing the probability curve for breaking strength in grams for certain fibres: the method of moments, the method of maximum likelihood and the method of minimal Chi-square, of which the latter two are shown to be closely related. By a method of successive approximations (carried to three steps) using ten-place logarithms, curves are obtained, which for the last two methods practically coincide, but which show readily observable discrepancy from that for a recalculated Pearsonian method of moments. The method of obtaining successive approximations consists in calculating $dm/d\theta$ and $(1/m)dm/d\theta$ for each of four parameters, and for each frequency class, and thence computing the discrepancies for the likelihood and for the minimal Chi-square. Applying corrections by Fisher's c-matrix method an adjusted curve is obtained as next approximation showing diminished discrepancies. *A. A. Bennett.*

von Schelling, H. Kennzeichen für eine rein zufällige Folge der Werte in einer zeitlich geordneten Beobachtungsreihe. Astr. Nachr. **269**, 155–159 (1939). [MF 1494]

The criteria for randomness due to Kermack and McKendrick [Proc. Roy. Soc. Edinburgh **57**, 228–240 (1937)] are first discussed. The results of these authors are supplemented by the calculation of the exact values of the second, third and fourth central moments of the distribution of runs and of the distribution of gaps. With these results in view, it is recommended for both runs and gaps that the difference between the observed sample mean and the mean on

the hypothesis of randomness divided by the square root of the estimate of the variance obtained from the sample be referred to Fisher's t-distribution for a test of significance, though no argument is given that this ratio actually obeys a t-distribution law, even approximately. Attention is then called to a test due to the author [Deutsch. Statist. Zentralblatt **26**, 137 (1934)], which does not require the condition that no two consecutive members of the series be equal, and which has recently been rediscovered by M. Kendall, S. Kendall and Smith [Biometrika **30**, 271–273 (1938–1939)]. Here a similar use of Fisher's t for a test of significance is suggested. A numerical example illustrating the application of the two types of tests is given.

C. C. Craig (Ann Arbor, Mich.).

✶Tintner, Gerhard. The Variate Difference Method. Monograph No. 5, Cowles Commission for Research in Economics. Principia Press, Inc., Bloomington, Ind., 1940. xiii+175 pp. $2.50.

In theoretical analysis of a statistical time series, say x_t $(t=1, 2, \cdots, n)$, the starting point can be chosen among many alternative hypotheses. An assumption frequently used is that the series is additively composed of a functional and a random element, say $x_t=f_t+r_t$. The variate difference method, first introduced by British statisticians and later developed mainly by O. Anderson [see, for example, Die Korrelationsrechnung in der Konjunkturforschung, Bonn, 1929], is chiefly used in that particular case of this hypothesis where the serial coefficients of the random element are zero (that is, $E(r_tr_u)=E(r_t)E(r_u)$, where $t\neq u$, and $E(r)$ denotes the expectation of r). Considering a series of differences, say $\Delta^{(k)}x_t=\Delta^{(k)}f_t+\Delta^{(k)}r_t$ $(t=1, 2, \cdots, n-k)$, the essential idea of the method is that the two components behave differently as k is allowed to increase; for polynomials, harmonics and other "smooth" elements f_t the differences are gradually reduced so that $\Delta^{(k_0)}f_t\sim 0$ for a sufficiently large k_0, while the differences of r_t tend to diverge. Consequently, $\Delta^{(k_0)}x_t\sim\Delta^{(k_0)}r_t$, and, since the assumptions mentioned give a simple relation between the variances $\sigma^2(\Delta^{(k)}r_t)$ and $\sigma^2(r_t)$, the latter can be estimated from the difference series $\Delta^{(k)}x_t$ $(t=1, 2, \cdots, n-k)$ if the order k is large enough. In this way each trial k gives an estimate of $\sigma^2(r_t)$, say σ_k^2. Under the hypotheses made, σ_k^2 is a decreasing sequence whose limit equals $\sigma^2(r_t)$. In practical applications, the difficult point is to decide how large k_0 must be taken in order that $\sigma_{k_0}^2$ might be regarded as a reasonable estimate of $\sigma^2(r_t)$. For use in this connexion, O. Anderson [loc. cit.] gives a criterion based on the variance of $\sigma_k^2-\sigma_{k+1}^2$.

The book contains 10 chapters and 7 appendices. The matter is presented from the economic point of view. Chapters II, III give a survey of the literature, including references to the criticism of the method. Chapter IV is devoted to fundamentals in the theory of probability and the calculus of finite differences. Chapters V–VII give a presentation of the method as outlined above. In accordance with a preliminary note [Ann. Math. Statistics **10**, 139–143] the author develops in Chapter VIII an alternative method for determining the order k_0. The method attaches to the sampling theory of R. A. Fisher and deals with the ratio $\sigma_k^2/\sigma_{k+1}^2$. The difficulty that σ_k^2 and σ_{k+1}^2 are intercorrelated is avoided by calculating each of these variances from a series of selected differences. As stated by the author, the resulting independence of the variances is bought at the price of a reduction of the information available. In Chapter IX, the Sheppard graduation method is used for the elimination of the random element r_t. In Chapter X, the variate difference method is employed to study the correlation between two time series. The book includes several tables which facilitate the calculations. Detailed illustrations of the application of the methods are given on the basis of four wholesale price series. The formal developments are concentrated to the appendices, which also contain a summary, a treatment of special problems, and a discussion of alternative hypotheses and methods in the analysis of time series.

H. Wold (Stockholm).

Applications of the Theory of Probability, Economics

Haldane, J. B. S. The equilibrium between mutation and random extinction. Ann. Eugenics **9**, 400–405 (1939). [MF 1461]

In his "The Genetical Theory of Natural Selection" [Oxford, 1930, chap. 4] R. A. Fisher calculated the frequencies of genes which are found in r loci of a population of N individuals, when mutation, selection and random extinction affect the composition of the population. Fisher assumed that the probability that a gene found in one locus of an individual will be represented by a similar gene in r of his descendants is given by Poisson's formula. According to the author this assumption is unsatisfactory for organisms where the descendants form large groups with highly correlated probabilities of surviving. The present paper, therefore, relaxes Fisher's hypothesis.

W. Feller.

Rhodes, E. C. Population mathematics. I. J. Roy. Statist. Soc. **103**, 61–89 (1940). [MF 1941]

Let $\phi(x)$ be the density of probability, at birth, of a female child giving birth, at an age between x and $x+dx$, to a female child. Then $\phi(x)$ is zero for all values of x outside the reproductive period $l<x<L$. The author studies the density $B(t)$ of female births at time t in a population originating from a female child born at time $t=0$. Obviously $B(t)=\int_l^L B(t-x)\phi(x)dx$, and since $B(t)=\phi(t)$ for $l\leqq t\leqq \min[2l, L]$, one can successively compute $B(t)$ in the intervals between the multiples of l and L. The author also treats the integral equations by methods which Lotka has introduced for this and similar problems.

W. Feller.

Feller, Willy. On the logistic law of growth and its empirical verifications in biology. Acta Bioth. Ser. A. **5**, 51–66 (1940). [MF 1704]

The logistic curve exploited by Pearl and others is often accepted as expressing a fundamental law of biological growth and this view (frequently plausible in special cases) seems to find support in the success that has attended the effort at fitting a logistic curve to various data on growth of experimental populations. Instead of being content with achieving a graphical vehicle providing satisfactorily close fit, some writers have sought to infer the necessary working of an autocatalytic reaction and hence to assume that the observed and tabulated growth is controlled by some internal factor resident in the organism, rather than being largely governed by external factors. The author considers the problem as to whether there is evidence to support the universal application of any such reasoning as to the operative causes. Given the general type of ogival curve bounded

by horizontal asymptotes, the author finds that, with three parameters at one's disposal (as in the case of the logistic curve), comparably satisfactory fitting may be achieved for curves of notably different analytic form, some showing skew-symmetry and others not. While extensive projects might perhaps be planned and carried out which could be legitimately regarded as providing some evidence for or against a theoretic logistic form in a given application, there seems no reason to accept currently available data as providing crucial a posteriori evidence on this question.

A. A. Bennett (Providence, R. I.).

Backman, Gaston. Methodik der theoretischen Wiedergabe beobachteter Wachstumsserien. Lunds Univ. Årsskrift **35**, no. 8, 20 pp. (1939) = Fysiogr. Sällskapets Handlingar **50**, no. 8, 20 pp. (1939). [MF 1971]

The author formulates a universal law of biological growth by the differential equation $y'=\exp[a+b\log t-c\log^2 t]$, where a, b, c are constants. It is shown that the solution is the ordinary probability function of an intermediary variable x, which in turn is of the form $A\log t+B$. The author also indicates a method for the practical computation and gives several examples.

W. Feller (Providence, R. I.).

Wald, A. The approximate determination of indifference surfaces by means of Engel curves. Econometrica **8**, 144–175 (1940). [MF 1786]

Given the quantities of n commodities or goods, the author uses q as a vector with its coordinates the n given quantities. A real function $I(q)$ is supposed to measure the preference of an individual in the sense that if he prefers the set of quantities called q to the set called $\bar{q}$, then $I(q)>I(\bar{q})$; and if he has no preference, $I(q)=I(\bar{q})$. In the latter case, q and $\bar{q}$ are called "indifference sets." "The set of points q for which the indicator $I(q)$ has a constant value is called an indifference surface." A given income E is supposed to purchase n quantities q^i at prices p^i; and thus $E=\sum p^i q^i$. Supposing that the individual will maximize $I(q)$, each q^i is a function of E and the p^i. As E varies, a curve in n dimensional space is obtained, called an Engel curve. The author considers the special case where $I(q)$ is of the second degree, and hence the Engel curves are linear, thus determined by two points. For k price situations, $C_1, \cdots, C_k$ are the Engel curves; and v_i is the vector from q_0 to q_i. Theorem I. If there exists an indicator $I(q)$ which is a polynomial of the second degree in q over the $(2k-1)$-dimensional linear space determined by the vectors $v_1, \cdots, v_{2k-1}$ associated with Engel curves $C_1, \cdots, C_k$, then the indicator is uniquely determined in S by the said Engel curves, apart from an arbitrary proportionality factor and an arbitrary additive constant. Theorems II–VIII elaborate the theory with the introduction of integrability conditions and the rank of the quadratic from $I(q)$; and Theorem IX expresses $I(q)$ in terms of "distinguished" coordinates.

E. L. Dodd (Austin, Tex.).

Ackermann, Wolf-Günter. Eine Erweiterung des Poissonschen Grenzwertsatzes und ihre Anwendung auf die Risikoprobleme in der Sachversicherung. Schr. Math. Inst., Inst. Angew. Math. Univ. Berlin **4**, 211–255 (1939). [MF 1516]

Consider a distribution function $F(x)$ and put $F_0(x)=0$ for $x<0$, $F_0(x)=1$ for $x\geqq 1$ and $F_{k+1}(x)=F_k(x)*F(x)$. It is known [Khintchine, Asymptotische Gesetze der Wahrscheinlichkeitsrechnung, Ergebnisse der Mathematik 2, 4 (1933)] that, for $t>0$ and $\lambda>0$,

$$W(x,t)=e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}F_k(x) \qquad (1)$$

is a distribution function, occurring in certain discontinuous stochastic processes. The author calls (1) a generalized Poisson-distribution. He first studies the simplest ways of computing (1) practically. Following, then, the general ideas of the collective risk-theory in life-insurance (F. Lundberg, Cramér) the author applies the theory of stochastic processes, that is to say, functions of type (1), to the theory of accident and fire insurance. The word limit-law in the title seems to stand simply for distribution.

W. Feller.

Fischer, Kurt. Beiträge zur Tarifanalyse. Bl. Versich.-Math. **4**, 491–506 (1939). [MF 1422]

Assuming that the known office-premiums are based on Höckner's formula, the author discusses the determination of the life table and the loadings used in computing the premiums. He lets α be the initial expenses, β the charges for collecting the premiums, γ the management expenses. The office-premium for age x and duration n is then given by

$$P_{x,n}=\frac{1+\alpha-(d-\gamma)a_{x,\overline{n}|}}{(1-\beta)a_{x,\overline{n}|}}=A\frac{1}{a_{x,\overline{n}|}}-B,$$

where $a_{x,\overline{n}|}$ is the temporary annuity, d the discount, $A=(1+\alpha)/(1-\beta)$ and $B=(d-\gamma)/(1-\beta)$. The ratio

$$\frac{P_{x,n}-P_{x+h,n}}{P_{x+h,n}-P_{x+2h,n}}=\frac{a_{x+h,\overline{n}|}-a_{x,\overline{n}|}}{a_{x+2h,\overline{n}|}-a_{x+h,\overline{n}|}}\cdot\frac{a_{x+2h,\overline{n}|}}{a_{x,\overline{n}|}}$$

is therefore independent of the loadings. By comparing the values of this expression, calculated for various life tables, with the ratio

$$\frac{P_{x,\overline{n}|}-P_{x+h,\overline{n}|}}{P_{x+h,\overline{n}|}-P_{x+2h,\overline{n}|}}$$

computed by means of the office-premiums, one is able to determine the life table. Instead of the last ratio for different ages, the author uses the single ratio

$$\frac{\sum_0^4 P_{30+i,\overline{n}|}-\sum_0^4 P_{25+i,\overline{n}|}}{\sum_0^4 P_{45+i,\overline{n}|}-\sum_0^4 P_{30+i,\overline{n}|}}$$

as a "characteristical value" for the life table. This procedure means that it is possible for two life tables to give the same characteristical value. Having determined the life table, the author calculates the values α, β, γ.

E. Lukacs (New York, N. Y.).

Meissner, Walter. Das Zinsfussproblem bei der Leibrente. Bl. Versich.-Math. **4**, 467–491 (1939). [MF 1421]

The "interest problem" is the problem of finding an approximation for life annuities at the rate of interest δ' when only the numbers of commutation for the rate $\delta\neq\delta'$ are given. The author discusses the formulas of Weber [Bull. Trimestr. Inst. Actuaires Français, no. 101], Steffensen [Skand. Aktuarietidskr. **1918**], Meidell [Skand. Aktuarietidskr. **1918**], Poukka [Skand. Aktuarietidskr. **1923**], Palmquist [Skand. Aktuarietidskr. **1921**], Christen [Mitt.

Verein. Schweiz. Versich.-Math. **1930**] and Borch [Skand. Aktuarietidskr. **1933**]. *E. Lukacs* (New York, N. Y.).

Meidell, Birger. Zur Theorie und Praxis der Berechnung des effektiven Zinsfusses bei Anleihen. Skand. Aktuarietidskr. **1939**, 122–151 (1939). [MF 2003]

Kershner, Richard. Note on compound interest. Amer. Math. Monthly **47**, 196–198 (1940). [MF 1924]

The formula $s_t = s_0 e^{it}$ for compound interest is characterized by eight axioms. *W. Feller* (Providence, R. I.).

Lukacs, Eugen. Zur Theorie der Selekttafeln. Skand. Aktuarietidskr. **1939**, 223–236 (1939). [MF 1515]

NUMERICAL AND GRAPHICAL METHODS

Rocard, Y. Méthodes pratiques de résolution de l'équation du quatrième degré. Revue Sci. (Rev. Rose Illus.) **77**, 661–662 (1939). [MF 1680]

Johansen, Paul. Iteration von Funktionen zweier reellen Variabeln und einer komplexen Variabeln. Skand. Aktuarietidskr. **1939**, 101–113 (1939). [MF 2002]

Uhler, Horace S. Recalculation and extension of the modulus and of the logarithms of 2, 3, 5, 7 and 17. Proc. Nat. Acad. Sci. U. S. A. **26**, 205–212 (1940). [MF 1605]

The natural logarithms of 2, 3, 5, and 7 were computed to 330 decimal places. The first 273 places agree with the values obtained by J. C. Adams in 1887. (No previous verification had been published.) Values to 330 places are also given for $\log_e 10$, $\log_e 17$, the modulus $\log_{10} e$ and arc tangents of the reciprocals of the integers 577, 2449, 4999 and 8749. Arc tangents of 1/451 and 1/10081 are given to 215 decimal places. Additional places are given for the author's previously published values of e^{10}, e^{-10}, sin 10 and cos 10. *P. W. Ketchum* (Urbana, Ill.).

Abramowitz, Milton. Note on the computation of the differences of the Si(x), Ci(x), Ei(x) **and** $-$Ei$(-x)$ **functions.** Bull. Amer. Math. Soc. **46**, 332–333 (1940). [MF 1843]

If $\phi(x)$ denotes any of the functions $\mathrm{Ci}(x)-\log x$, $\mathrm{Ei}(x)-\log x$, $-\mathrm{Ei}(-x)+\log x$, it is shown that the second differences of $\phi(x)$ may be approximated by $h^2\phi''(x)$ to 12 decimal places, in the range $0<x<0.1$, with a tabular interval $h=10^{-4}$. Suitable approximations are given for $\phi''(x)$. *R. M. Foster* (New York, N. Y.).

Lowan, A. N. and Blanch, G. Tables of Planck's radiation and photon functions. J. Opt. Soc. Amer. **30**, 70–81 (1940). [MF 1155]

The functions with which these tables are concerned are

$$\mathcal{R}_\lambda = c_1\lambda^{-5}(e^{c_2/\lambda T}-1)^{-1}; \qquad \mathcal{R}_{0-\lambda} = \int_0^\lambda \mathcal{R}_\lambda d\lambda;$$

$$N_\lambda = 2\pi c\lambda^{-4}(e^{c_2/\lambda T}-1)^{-1}; \quad N_{0-\lambda} = \int_0^\lambda N_\lambda d\lambda,$$

where

$c = (2.99776 \pm 0.00020) \times 10^{10}$ cm sec.$^{-1}$ [the velocity of light],

$c_1 = (3.732 \pm 0.006) \times 10^{-5}$ erg cm^2 sec.$^{-1}$ [$=2\pi c^2 h$, where h is Planck's constant of action],

$c_2 = (1.436 \pm 0.001)$ cm K° [$=ch/k$, where k is Boltzmann's constant].

$\mathcal{R}_{0-\lambda}$, $N_{0-\lambda}$ express, respectively, the rates of emission of energy and photons per unit area per unit time.

Table II gives the functions $\mathcal{R}_{0-\lambda}$, $N_{0-\lambda}$, $\mathcal{R}_\lambda$, N_λ for $T=1000°$ K with argument λ ranging from 0.50 to 20 microns. Table I gives the ratio of each of the above functions to its maximum value with argument λT cm K° in the range 0.05 to 2.00. Table III gives N_λ with argument λ in the range 0.25 to 10 for the following values of T; 1000, 1500, 2000, 2500, 3000, 3500, 6000° K. Means of correction for changes in c_1, c_2 are provided, and in the case of Tables I and II printed first and second differences are provided with appropriate interpolation formulae. *L. M. Milne-Thomson* (Greenwich).

Weinel, E. Eine Erweiterung des Grammelschen Verfahrens zur Berechnung von Eigenwerten und Eigenfunktionen. Ing.-Arch. **10**, 283–291 (1939). [MF 1393]

The method of Grammel, a variant of that of Rayleigh-Ritz, is considered the first step of iterations, essentially those of Kellogg. Resulting approximations of eigen-values are investigated. The relative accuracies of this and similar methods are compared. *K. Friedrichs.*

Myers, D. M. An integraph for the solution of differential equations of the second order. J. Sci. Instruments **16**, 209–222 (1939). [MF 1621]

A portable integraph is described for providing a continuous solution of

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy+d=0,$$

where b and c are constants, including zero, and a and d are given functions of x, y or dy/dx. Limitations placed by the construction of the instrument on the form, range and signs of these coefficients frequently may be circumvented by transforming variables. For given initial conditions, the solution is provided in the form of a plot of y and dy/dx against x. The instrument sacrifices flexibility for simplicity in construction and operation, but is capable of solving easily and quickly and with engineering accuracy a wide range of equations. The speed of the instrument allows its effective use in solving by trial and error cases of incomplete initial conditions and given terminal conditions, and the case of steady state. The instrument is of the rolling type and employs two integrating mechanisms of the type invented by Abdank-Abankanowicz and used by Coradi. The values of the terms in the equation are represented by linear displacements of linkages. Given functions of the variables are introduced from plots. *S. H. Caldwell.*

Killian, K. Planimeter-Studie. Allg. Vermessgs-Nachr. **51**, 666–671 (1939). [MF 1401]

The "null-curves" of a polar planimeter are the curves traced out by the pointer when the pointer is moved in such

a way that the rolling wheel does not rotate. The author determines the equations of these curves and gives a diagram of typical ones. With the exception of one, the "null-circle," they are all spirals which are asymptotic to this null-circle. Applying this idea to the practical use of the planimeter in measurements of areas, it is shown that the least error occurs when one starts at a point on the given curve where the tangent is parallel to a null-curve and with the pointer arm perpendicular to the polar arm. Also, a small long rectangle should have its long side perpendicular to the null-curve. *P. W. Ketchum* (Urbana, Ill.).

Maxwell, Louis R. An electrical method for compounding sine functions. Rev. Sci. Instruments **11**, 47–54 (1940). [MF 1146]

A harmonic synthesizer of two elements has been developed to facilitate studies in electron diffraction. An alternating supply voltage is modulated by variometers of special design which are driven at harmonic frequencies by a gear train. The combined outputs are detected by a vacuum tube voltmeter and photographically recorded from a galvanometer mirror. Precision in the location of maxima or minima is 0.03 or 0.04 radians. The author includes a comprehensive bibliography on computing devices. *R. L. Dietzold.*

Espley, D. C. Harmonic analysis by the method of central differences. Philos. Mag. **28**, 338–352 (1939). [MF 1118]

In the physical problem of the distortion produced by thermionic valves, the spurious harmonics may be calculated by substituting the imposed fundamental $e=k\sin\omega t$ in the equation of the load characteristic $I=f(e)$. The mathematical problem is the following: The function $f(e)$ is given empirically by a set of observations. If $f(e)$ is approximated by a polynomial $P(e)$, then $P(k\sin\omega t)$ will have the form

$$a_0+a_1\sin\omega t+a_2\cos 2\omega t+a_3\sin 3\omega t+\cdots,$$

where the coefficients are polynomials in k which depend on the given data. The problem is to express the a's directly in terms of the given data. To do this, the author uses the Newton-Stirling central difference formula to give a polynomial which interpolates to the data. The a's then become linear combinations of central differences of the data, the coefficients in these linear combinations being polynomials $p_n(k)$ in k. The $p_n(k)$ are tabulated for ten values of k for $a_0, \cdots, a_3$. *P. W. Ketchum* (Urbana, Ill.).

Schumann, T. E. W. The principles of a mechanical method for calculating regression equations and multiple correlation coefficients and for the solution of simultaneous linear equations. Philos. Mag. **29**, 258–273 (1940). [MF 1667]

Dr. Schumann describes a machine with the following parts. For n independent variables, n horizontal beams are set each to rotate about its longitudinal axis. From these extend adjustable arms at intervals along the beam, all in one plane; and at the beginning of an operation this plane is set horizontal. The length of the arms projecting from one side of a beam represent positive numbers; from the other side negative numbers all associated with one of the independent variables. An inextensible cord starting from above passes under the extremity of an arm, thence to a pulley above, now down to the corresponding arm for the next variable, and so on, and is finally made to suspend a rod in a mercury bath. Thus, one rod is connected to $y_1, z_1, \cdots$; the next rod to $y_2, z_2, \cdots$; etc.; and all rods are immersed in the same bath. A mathematical explanation of the machine is given. The author estimates that, for four independent variables, with his machine computation of multiple correlation coefficients can be effected in one-fourth the usual time; for ten variables, in 2/25 the usual time. Preliminary experiments with a rough small model gave results correct to within 7 per cent. He compares his machine with those of Mallock, Vidal and Wilbur. *E. L. Dodd.*

Nomography

Grammel, R. Ein Gegenstück zum Meissnerschen Verfahren der graphischen Analysis. Ing.-Arch. **10**, 395–411 (1939). [MF 1191]

A function $p(\theta)$ is represented graphically by its "polar map," which is the curve formed by points with polar coordinates $(1/p, \theta)$. The "orthopolar" of this curve is the polar map of $p'(\theta)$; it may be obtained by extending the tangents to the polar map until they intersect radius vectors at angles $\theta+\pi/2$. Consequently, there is a very simple graphical means for differentiation; and the process may be reversed for graphical integration. The method is readily extended to give graphical solutions of any ordinary differential equation with one point boundary conditions. As examples, scale drawings are given for the solution of the differential equations arising in the following physical problems: (1) The shape of a bridge suspension cable. (2) The bending of a slice of a rotating turbine (the two point boundary conditions are first reduced to one point conditions). (3) The rigidity of a hollow shell which has a cross section and load symmetric about an axis of rotation. (4) The trajectory of a high-altitude projectile. *P. W. Ketchum* (Urbana, Ill.).

Frank, M. L. Nomogramme pour l'équation complète du quatrième degré. Uchenye Zapiski Moskov. Gos. Univ. Nomografija **28**, 153–154 (1939). (Russian) [MF 1698]

L'auteur construit un nomogramme à transparent orienté pour l'équation complète de l'ordre 4: $z^4+a_3z^3+a_2z^2+a_1z+a_0=0$. Le plan du nomogramme contient deux échelles des variables a_1 et a_2 et une famille de courbes a_0. Le transparent porte une famille de courbe a_3 et un faisceau de droites a_3. L'abscisse du point d'intersection de la courbe a_0 du plan du nomogramme avec la courbe a_3 du transparent donne le valeur de z. Une construction analogue peut être appliquée pour l'équation réduite du cinquième degré. *N. A. Glagoleff* (Moscou).

Gorbounov-Possadov, M. I. Sur un type d'équations du cinquième ordre nomographique. Uchenye Zapiski Moskov. Gos. Univ. Nomografija **28**, 21–26 (1939). (Russian) [MF 1688]

L'auteur construit plusieurs types de nomogrammes pour l'équation non-anamorphosable du cinquième ordre (1) $f_3=\phi_1\phi_2+\psi_1\psi_2$, entre autres, un nomogramme qui donne une généralisation du nomogramme à index parallèle: outre d'un index parallèle on est oblige de se servir d'un compas pour marquer les points sur les échelles. En comparant ensuite l'équation (1) à la formule de l'aire du triangle un des sommets duquel est à l'origine, l'auteur reduit le problème à un calcul graphique de l'aire du triangle. Enfin

l'auteur construit un nomogramme à trois dimensions qu'il reduit ensuite à un nomogramme plan.

O. W. Ermolowa (Moscou).

Mandzuk, A. I. Nomogrammes à systèmes de points non-côtés. Uchenye Zapiski Moskov. Gos. Univ. Nomografija **28**, 71–74 (1939). (Russian) [MF 1692]

L'auteur construit pour l'équation

$$\left|\frac{1}{\varphi_i(\alpha_i)}\ \frac{1}{\psi_i(\beta_i)}\ 1\right|=0, \qquad i=1, 2, 3,$$

un nomogramme suivant: les échelles des variables α_i sont tracées sur un des axes de coordonnées (ou sur des droites parallèles à cet axe); ceux des variables β_i sur l'autre axe. Pour trouver la valeur d'une des variables l'auteur se sert des trois droites passant par un même point et joignant des points correspondant des échelles (α_i) et (β_i), $i=1, 2, 3$. En produisant une transformation correlative du nomogramme, l'auteur obtient un nouveau type de nomogrammes pour la même équation. Un nomogramme plus compliqué peut être construit pour une équation non-anamorphosable à six variables. *O. W. Ermolowa* (Moscou).

Gorodsky, M. Sur la construction approximée des nomogrammes à points alignés pour une équation quelconque. Uchenye Zapiski Moskov. Gos. Univ. Nomografija **28**, 15–19 (1939). (Russian) [MF 1687]

L'auteur donne une méthode de construction approximée des nomogrammes à points alignés pour une équation quelconque à trois variables en se servant des approximations successives. Étant donnée une équation (1) $z=F(x, y)$, il choisit arbitrairement deux valeurs x_1 et x_2 de la variable x et deux valeurs y_1 et y_2 de la variable y; ensuite il construit une échelle arbitraire (z') de la variable z. En déterminant z_1 et z_2 à l'aide des conditions $z_1=F(x_1, y)$ et $z_2=F(x_2, y)$, l'auteur construit les points de l'échelle $\underline{(y)}$ en cherchant les points d'intersection des droites $\overline{x_1z_1}$ et $\overline{x_2z_2}$. L'échelle (x) est construite analogiquement. Le nomogramme ainsi obtenu n'est exact que dans les deux cas suivants: (1) $x=x_1$, ou $x=x_2$, y étant arbitraire, (2) $y=y_1$, ou $y=y_2$, x étant arbitraire. Ensuite l'auteur construit l'échelle (z''); il trouve des paires de valeurs des variables x et y, qui substituées dans l'équation (1) donnent des valeurs de z à peu près égales entre elles. Il mène les droites par les points correspondants des échelles (x) et (y); ces droites forment un faisceau de droites passant à peu près par une même point. En menant une courbe par les parties les plus étroites des faisceaux, l'auteur obtient l'échelle (z'') (première approximation). En laissant invariables les échelles (z'') et (x), il reconstruit l'échelle (y); puis à l'aide des nouvelles échelles (z'') et (y'), l'échelle (x') et enfin obtient l'échelle (z''') (seconde approximation) et ainsi de suite. Dans le cas d'une équation anamorphosable on reçoit à la suite de plusieures approximations un nomogramme exact à points alignés.

O. W. Ermolowa (Moscou).

Pentkowsky, M. W. La transformation projective des nomogrammes. Uchenye Zapiski Moskov. Gos. Univ. Nomografija **28**, 115–140 (1939). (Russian) [MF 1695]

L'auteur donne une méthode générale pour la recherche des transformations projectives, servant à modifier la forme d'un nomogramme déjà construit. Dans la première partie il considère tous les types des transformations collinéaires du plan et choisit des sous-groupes du groupe général de collinéations; il donne la classification de ces sous-groupes selon les éléments invariables du plan. La collinéation étant déterminée par les éléments invariables, l'auteur trouve dans chaque groupe de telles collinéations un sous-groupe à un paramètre et son trajectoire. Cela permet d'observer la variation de la collinéation avec la variation des paramètres qui la déterminent. Aux divers types de mouvement des points dans le plan l'auteur fair correspondre divers types de mouvements dans le plan non-euclidien. À l'aide de cette interprétation des transformations collinéaires l'auteur fait correspondre à chacun des types de collinéations un réseau formé par deux familles de courbes qui sont les trajectoires des sous-groupes à un paramètre du groupe donné. Dans la deuxième partie l'auteur utilise les réseaux obtenus pour la transformation des nomogrammes. Il montre qu'en posant le nomogramme sur un tel réseau et en le faisant glisser sur le réseau on peut determiner de telles valeurs des paramètres de la collinéation qui font varier la transformation du nomogramme dans le sens voulu.

N. A. Glagoleff (Moscou).

Pentkowsky, M. W. Les nomogrammes plans équivalents aux nomogrammes à trois dimensions de Mehmke. Uchenye Zapiski Moskov. Gos. Univ. Nomografija **28**, 141–148 (1939). (Russian) [MF 1696]

Les nomogrammes de Mehmke sont formés de quatre échelles situées sur des courbes gauches; les côtes des quatre points, situés dans le même plan, vérifient l'équation

$$(1) \qquad |1\phi_i\psi_i\xi_i|=0, \qquad \text{où } i=1, 2, 3, 4.$$

L'auteur construit un nomogramme plan qui peut remplacer le nomogramme à trois dimensions de Mehmke; ce nouveau nomogramme consiste en six systèmes de points à deux côtes de variables (z_1, z_2); (z_1, z_3); (z_1, z_4); (z_2, z_3); (z_2, z_4); (z_3, z_4). Il obtient les familles de courbes de chaque système comme une intersection du plan du nomogramme plan avec des familles de cônes dont les sommets sont situés sur l'une des échelles curvilignes et pour lesquels l'autre échelle sert de directrice. Comme les six points dont les côtes sont liées par la relation (1) sont situés sur une droit, on peut supprimer deux ou trois des six systèmes de points à deux côtes. Si l'une des échelles est plane, on peut réduire une des systèmes à une échelle en choisissant le plan du nomogramme parallèle au plan de cette échelle. Si deux des échelles sont situées dans le même plan, deux systèmes se réduisent à des échelles. Si toutes les quatre échelles sont planes, situées deux par deux dans de mêmes plans, on obtient un nomogramme à double alignement.

O. W. Ermolowa (Moscou).

Moldawer, A. I. Les nomogrammes hexagonaux de Lallemann. Uchenye Zapiski Moskov. Gos. Univ. Nomografija **28**, 107–114 (1939). (Russian) [MF 1694]

L'auteur établie la correspondance entre un nomogramme hexagonal et l'abaque cartésien; il démontre que le nomogramme de Lallemann n'est qu'un cas particulier d'un nomogramme plus général: (1) les trois échelles peuvent ne pas concourir par un même point; (2) les angles que forment les échelles (ainsi que les droites du transparent) peuvent être arbitraires; (3) les droites du transparent couper les droites du nomogramme sous des angles arbitraires. En outre l'auteur montre qu'une inversion transforme le nomogramme hexagonal en un nomogramme à points alignés à échelles concourantes. Enfin l'auteur montre qu'on peut toujours modifier les angles entre les droites du nomogramme afin d'éviter les intersections obliques des droites du transparent avec celles des échelles. Cette généralisation

permet de remplacer les modules invariables des échelles de Lallemann par des modules qui dépendent des angles entre les échelles; cela permet à élargir le domaine de l'application des nomogrammes hexagonaux. *O. W. Ermolowa.*

Bütner, H. A. Sur le problème de l'anamorphose générale. Uchenye Zapiski Moskov. Gos. Univ. Nomografija **28**, 7–13 (1939). (Russian) [MF 1686]

L'auteur envisage un cas particulier de l'anamorphose générale: il donne les conditions nécessaires et suffisantes que doit vérifier la fonction $F(x, y, z)$ pour que l'équation (1) $F(x, y, z)=0$ puisse être représentée par un nomogramme du genre 1. Prenant l'équation (1) sous la forme

$$(2)\qquad \begin{vmatrix} f_1 & 0 & 1 \\ f_2 & g_2 & 1 \\ f_3 & 1 & 1 \end{vmatrix} = 0,$$

il élimine les fonctions f_1, f_2, g_2 et f_3 de l'équation (2) à l'aide de différentiations successives et obtient deux relations entre les dérivées partielles de divers ordres de z par rapport aux variables x et y. Ces relations sont les conditions cherchées; leur nécessité suit de la méthode même à l'aide de laquelle elles étaient déduites; leur suffisance doit être démontrée par l'intégration des équations trouvées: l'auteur obtient l'équation qui lie les variables x, y et z et qui peut être représentée par un nomogramme du genre 1.

N. A. Glagoleff (Moscou).

Ermolowa, O. W. Sur la dissociation des variables dans une équation à un nombre quelconque de variables. Uchenye Zapiski Moskov. Gos. Univ. Nomografija **28**, 43–54 (1939). (Russian) [MF 1690]

L'auteur donne les conditions nécessaires et suffisantes dans lesquelles l'équation à n variables (1) $F(x_1, x_2, \cdots, x_n)=0$ peut être remplacée par un système de $n-2$ équations des types suivants:

$$(2)\qquad \phi(x_r, x_s, \alpha_q)=0;\quad \psi(x_l, \alpha_g, \alpha_h)=0;\quad \chi(\alpha_u, \alpha_v, \alpha_w)=0,$$

$\alpha_s, \alpha_g, \cdots$ étant $n-3$ variables auxilières. Dans ce cas l'équation (1) est représentable par un nomogramme composé (à systèmes ramifiés). L'auteur démontre d'abord le théorème fondamental: Pour que l'équation (1) puisse être mise sous la forme:

$$(3)\qquad Q(x_1, x_2, \cdots, x_m)=\alpha;\quad H(x_{m+1}, x_{m+2}, \cdots, x_n)=\alpha,$$

α étant une variable auxilière et $2\leqq m\leqq n-2$, il faut et il suffit qu'aient lieu les conditions:

$$(4)\qquad \begin{aligned} p_1 : p_2 : \cdots : p_m &= p_{1,m+1} : p_{2,m+1} : \cdots : p_{m,m+1} \\ &= p_{1,m+2} : p_{2,m+2} : \cdots : p_{m,m+2} = \cdots \\ &= p_{1,n-1} : p_{2,n-1} : \cdots : p_{m,n-1}, \end{aligned}$$

où

$$p_i=\partial x_n/\partial x_i,\quad p_{ik}=\partial^2 x_n/\partial x_i\partial x_k.$$

Dans le cas $n=4$ les conditions (4) se réduisent à la condition bien connue de Goursat. La démonstration de la suffisance des conditions (4) qui consiste dans l'intégration des systèmes des équations aux dérivées partielles du deuxième ordre (4) renferme ainsi la méthode pour produire la dissociation des variables, c'est-à-dire, pour déterminer les fonctions Q et H. La même méthode s'applique à chacune des équations (3) (dans le cas où cela est possible) et le procédé de la dissociation se prolonge jusqu'à ce qu'on obtienne un système d'équations telles que (2). Le système (2) peut être représenté à l'aide d'un schème où chacune des équations à trois variables est représentée par un point et par les trois droites issues de ce point. À l'aide du schème on peut obtenir directement les conditions de la dissociation complète des variables dans l'équation (1) sans produire des dissociations consécutives. *N. A. Glagoleff.*

Denissuk, U. N. Les méthodes analytiques de la corrélation approximée et les problèmes fonctionnels correspondants. Uchenye Zapiski Moskov. Gos. Univ. Nomografija **28**, 27–42 (1939). (Russian) [MF 1689]

L'auteur donne des méthodes nouvelles pour la construction approximée des nomogrammes des équations non-anamorphosables. Soit (1) $F(x, y, z)=0$ une telle équation. En envisageant z comme fonction de x et y, définie par (1), l'auteur cherche la valeur minimale de l'intégrale

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\begin{vmatrix} f_1 & g_1 & 1 \\ f_2 & g_2 & 1 \\ f_3 & g_3 & 1 \end{vmatrix}^2 dxdy,$$

f_i et g_i étant des fonctions qui déterminent les échelles du nomogramme. L'auteur résout ce problème pour des nomogrammes à échelles parallèles. Dans ce cas les fonctions g_1, g_2 et g_3 se réduisent à des constantes. En donnant a priori la graduation de l'échelle (z), l'auteur détermine les fonctions f_1, f_2 en égalant à zéro la variation de l'intégrale

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\{f_1(x)+f_2(y)-\omega(x, y)\}^2 dxdy,$$

la fonction $\omega(x, y)=f_3(z)$ étant déterminée par l'équation (1). Pour déterminer les fonctions $f_1(x)$ et $f_2(y)$ l'auteur obtient deux équations intégrales dont la solution est élémentaire. Ainsi il obtient une expression approximée suivante pour la fonction $\omega(x, y)$:

$$\omega(x, y)\approx\frac{\int_{y_1}^{y_2}\omega(x, t)dt}{y_2-y_1}+\frac{\int_{x_1}^{x_2}\omega(t, y)dt}{x_2-x_1}-\frac{\int_{x_1}^{x_2}\int_{y_1}^{y_2}\omega(t, \tau)dtd\tau}{(x_2-x_1)(y_2-y_1)}.$$

Dans le cas des nomogrammes plus compliqués l'auteur tâche de représenter l'équation (1) sous la forme suivante: $\Omega(x, y, z)+\omega(x, y)=0$, Ω étant une fonction anamorphosable et $\omega(x, y)$ une fonction n'ayant qu'une influence bornée sur les valeurs de z. L'auteur donne $\Omega(x, y, z)$ sous la forme de Cauchy: $\Omega(x, y, z)=F(x)+G(x)H(y)+G_1(x)H_1(z)$, et la fonction $\omega(x, y)$ ou sous la forme $H(y)g(x)+f(x)$ ou sous la forme $G(x)h(y)+f(x)$. Dans le premier cas l'échelle curviligne seule est déformée; dans le second cas c'est en outre la graduation de l'échelle (y). Dans le premier cas l'auteur détermine les fonctions $f(x)$ et $G(x)$ en égalant à zéro la variation de l'intégrale

$$\iint_G\{f(x)+g(x)H(y)-\omega(x, y)\}^2 dxdy,$$

G étant le domaine de la variation des variables x et y. Dans le second cas l'auteur détermine les fonctions $f(x)$ et $h(y)$ en égalant à zéro la variation de l'intégrale

$$\iint_G\{f(x)+h(y)G(x)-\omega(x, y)\}^2 dxdy.$$

L'auteur donne la solution des deux problèmes cités et montre qu'il est possible de les généraliser pour le cas du plusieurs variables, dans celui d'un nomogramme à systèmes de points à deux côtes et dans le cas d'un nomogramme du genre 2, résolvant une équation d'ordre nomographique 5. La résolution du dernier problème mène à un système de deux équations intégrales à noyau symétrique de Schmidt

et ensuite à une équation de Fredholm de la deuxième espèce. *N. A. Glagoleff* (Moscou).

Sokoloff, B. D. Triangle à reverbération. Uchenye Zapiski Moskov. Gos. Univ. Nomografija **28**, 149–152 (1939). (Russian) [MF 1697]

L'auteur donne une généralisation du triangle des mélanges de Gibbs pour une équation à plusieures variables. Il construit six triangles équilatérales formant un hexagone, ce qui permet de résoudre le problème de la distribution des mélanges pour le cas de plusieurs ingredients. Ensuite il superpose les triangles les uns sur les autres; les lignes de liaison au lieu de coupes les côtés des triangles "reverbérant" des côtés successifs comme un rayon d'un miroir (d'où "triangle à reverbération"). Cette construction permet de diminuer les dimensions totales du nomogramme sans en diminuer l'exactitude. *O. W. Ermolowa.*

Ermolowa, O. W. L'échelle régulière sur une courbe unicursale et ses applications nomographiques. Uchenye Zapiski Moskov. Gos. Univ. Nomografija **28**, 55–70 (1939). (Russian) [MF 1691]

L'auteur donne une méthode géométrique de la construction d'une échelle "régulière" (dans le sens projectif) sur une quadrique unicursale C_4 dont un des points doubles ne coincide pas avec les deux autres. Parmi les points d'intersection de cette échelle avec une famille de coniques Σ_2 passant par deux des points doubles de C_4, quatre seulement sont mobiles; leurs côtes u, v, w, et t vérifient l'équation suivante: (1) $a_4uvwt+a_3(uvw+uvt+uwt+vwt)+a_2(uv+\cdots+wt)+a_1(u+v+w+t)+a_0=0$, les coefficients $a_0, a_1, \cdots, a_4$ étant les fonctions des côtes de deux des points doubles de C_4 désignés plus haut. Si ces côtes sont égales à 0 et à ∞, l'équation (1) se réduit à (2) $uvwt=a$. L'auteur construit pour cette équation un nomogramme à transparent orienté (ce qui est possible, si deux des points doubles de C_4 sont à l'infini). Les 4 échelles (u), (v), (w) et (t) du nomogramme ont pour support la même quadrique C_4. Sur le transparent est tracée une famille de coniques à un paramètre (les deux autres paramètres sont ceux de la translation). Ensuite l'auteur déduit l'équation que doivent vérifier les côtes de huit points mobiles de C_4 situés sur une cubique C_3, convenablement déterminée. Le même problème est résolu pour les côtes des points de C_4 situés sur une droite. Enfin l'auteur généralise la question pour une courbe unicursale C_n de degré n en donnant les relations entre les côtes des points de C_n situés sur une courbe de degré $n-1$, de degré $n-2$ et de degré $n-3$. *N. A. Glagoleff* (Moscou).

Moldawer, A. I. Les courbes unicursales et leur application à la nomographie. Uchenye Zapiski Moskov. Gos. Univ. Nomografija **28**, 75–106 (1939). (Russian) [MF 1693]

L'auteur considère les équations générales d'une échelle curviligne:

$$x=\frac{a_{10}f^n+a_{11}f^{n-1}+\cdots+a_{1n}}{a_{30}f^n+a_{31}f^{n-1}+\cdots+a_{3n}},\quad y=\frac{a_{20}f^n+a_{21}f^{n-1}+\cdots+a_{2n}}{a_{30}f^n+a_{31}f^{n-1}+\cdots+a_{3n}} \tag{1}$$

et étudie ses propriétés afin d'en simplifier la construction. Il donne le nom de "caractéristique" aux déterminant de l'ordre 3 qui forment la matrix de l'échelle

$$\left\|\begin{matrix} a_{10} & a_{11} & \cdots & a_{1n} \\ a_{20} & a_{21} & \cdots & a_{2n} \\ a_{30} & a_{31} & \cdots & a_{3n}\end{matrix}\right\|. \tag{2}$$

Les $3n-5$ de ces caractéristiques sont indépendantes. Elles déterminent l'échelle jusqu'à une transformation projective. L'auteur indique une méthode de construction de l'échelle quand les caractéristiques sont données et montre que les rapports de toutes les caractéristiques à l'une d'elles sont des invariants projectifs de l'échelle. Ensuite l'auteur construit trois échelles d'ordre n, formant un nomogramme à trois variables et donne la forme générale de l'équation pour laquelle un tel nomogramme peut être construit. Comme application de la théorie générale l'auteur envisage certains types d'équations à trois variables, ainsi que les équations à quatre variables pouvant être résolues par des nomogrammes formés d'échelles coniques à transparent: transparent à index parallèle, à index en equerre à index circulaires. En associant au nomogramme une échelle non côtée quelconque, l'auteur obtient l'équation la plus générale à quatre variables admettant un nomogramme à points alignés. La même méthode sert à étudier les échelles d'ordre trois et à obtenir les équations à trois et à quatre variables admettant des nomogrammes à transparents du même type. Enfin l'auteur envisage brièvement les propriétés des échelles du quatrième degré et donne quelques exemples de nomogrammes à points alignés à échelles de degré quatre et cinq. *N. A. Glagoleff* (Moscou).

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